Mathematics 204

Fall 2010

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam II consists of this cover page, 5 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

	1	2	3	4	5	6	**Sum**
points earned							
maximum points	17	17	16	16	17	17	100

$$y'' + y' + \tfrac{1}{4}y = \frac{e^{\cdots}}{4t^2}$$

$$\Updownarrow$$

1.[17] Find the general solution of $4y'' + 4y' + y = \dfrac{e^{-t/2}}{t^2}$ on the interval $t > 0$.

$y = e^{rt}$ in $4y'' + 4y' + y = 0$ leads to $4r^2 + 4r + 1 = 0$ so $(2r+1)^2 = 0$ and $r = -1/2$ (multiplicity two). Consequently, $y_1(t) = e^{-t/2}$ and $y_2(t) = te^{-t/2}$ form a fundamental set of solutions to $4y'' + 4y' + y = 0$ on any interval since

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t/2} & te^{-t/2} \\ -\frac{1}{2}e^{-t/2} & (1 - \frac{t}{2})e^{-t/2} \end{vmatrix} = (1 - \tfrac{t}{2})e^{-t} + \tfrac{t}{2}e^{-t} = e^{-t} \neq 0 .$$

Thus $y_c(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}$ is the complementary solution for the nonhomogeneous equation.

We use variation of parameters to find a particular solution to the nonhomogeneous equation:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)e^{-t/2} + u_2(t)te^{-t/2}$$

where

$$u_1 = \int \frac{-g\,y_2}{W}\,dt = \int \frac{-\frac{e^{-t/2}}{4t^2}\cdot te^{-t/2}}{e^{-t}}\,dt = -\frac{1}{4}\int \frac{1}{t}\,dt = -\frac{1}{4}\ln|t| + \cancel{C}^{\,0},$$

and

$$u_2 = \int \frac{g\,y_1}{W}\,dt = \int \frac{\frac{e^{-t/2}}{4t^2}\cdot e^{-t/2}}{e^{-t}}\,dt = \frac{1}{4}\int \frac{1}{t^2}\,dt = -\frac{1}{4t} + \cancel{C}^{\,0}.$$

Therefore, $y_p(t) = -\frac{1}{4}\ln(t)e^{-t/2} - \frac{1}{4t}\cdot te^{-t/2} = -\frac{1}{4}\ln(t)e^{-t/2} - \frac{1}{4}e^{-t/2}$ is a particular solution of the nonhomogeneous equation on the interval $t > 0$. Thus,

$$\boxed{y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2} - \tfrac{1}{4}\ln(t)e^{-t/2}} \qquad (c_1, c_2 \text{ arbitrary constants})$$

is the general solution of the nonhomogeneous equation on the interval $t > 0$.

Note: We absorbed the term $-\frac{1}{4}e^{-t/2}$ in the particular solution into the term $c_1 e^{-t/2}$ in the general solution since c_1 is an arbitrary constant.

2.[17] Consider the differential equation $y''' - y = 7e^t$

(a) Classify the differential equation by giving its order, stating whether it is linear or nonlinear, homogeneous or nonhomogeneous, and whether it has constant or variable coefficients. The DE is of third order, linear, nonhomogeneous, and has constant coefficients.

(b) Which of the following solution methods are valid for solving this differential equation? If the method is valid, give a potential drawback in using this method.

Method of Undetermined Coefficients: Valid; differentiating the trial particular solution and solving for cons

Variation of Parameters: Valid; need to evaluate four third-order determinants.

Laplace Transform: Valid; messy partial fraction decompositions.

(c) Find the general solution of the differential equation. The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ may be useful.

$y = e^{rt}$ in $y''' - y = 0$ leads to $0 = r^3 - 1 = (r-1)(r^2+r+1)$ so $r = 1$ or $r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Therefore $y_c(t) = c_1 e^t + e^{-t/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$ is the complementary solution. Since $g(t) = 7e^t$ we would normally use a trial particular solution of the form $Y(t) = Ae^t$. However, e^t is a solution to the associated homogeneous equation $y''' - y = 0$ so we must modify Y by multiplying by $t^s = t^1$. Therefore $y_p(t) = Ate^t$ is a trial solution to the nonhomogeneous equation; here A is a constant to be determined. Note that $y_p' = Ate^t + Ae^t$ by the product rule, so $y_p' = A(t+1)e^t$. Similarly, $y_p'' = A(t+2)e^t$ and $y_p''' = A(t+3)e^t$. We want $y_p''' - y_p = 7e^t$, so substituting we have $A(t+3)e^t - Ate^t = 7e^t$. Simplifying yields $3Ae^t = 7e^t$ so $A = \frac{7}{3}$. That is, $y_p = \frac{7}{3}te^t$ is a particular solution. Consequently, the general solution $y = y_c + y_p$ is

$$y(t) = \boxed{c_1 e^t + e^{-t/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) + \frac{7}{3}te^t}$$

where c_1, c_2, and c_3 are arbitrary constants.

3.[16] (Please use 32 ft/sec^2 as the acceleration of gravity in this problem.) A body weighing 8 pounds hangs from a vertical spring attached to the ceiling. At its equilibrium position, the body stretches the spring 1/2 ft from its natural length. The body is started in motion from the equilibrium position with an initial velocity of 4 ft/sec in the downward direction.

(a) Assuming that there is no damping and that the body is acted on by a downward external force of $3\cos(2t)$ pounds, set up, **BUT DO NOT SOLVE,** an initial value problem describing the motion of the body.

$mu'' + \gamma u' + ku = f(t)$ models the motion of the body where $\gamma = 0$, $f(t) = 3\cos(2t)$,

mg = weight so $m = \frac{8}{g} = \frac{8}{32} = \frac{1}{4}$ slug, and ku_0 = weight so $k = \frac{8}{u_0} = \frac{8}{1/2} = 16$ lb/ft.

Thus

$$\boxed{\frac{1}{4}u'' + 16u = 3\cos(2t), \quad u(0) = 0, \; u'(0) = 4,}$$

is an IVP modeling the vertical displacement $u(t)$ of the body from its static equilibrium position

(b) If the given downward external force is replaced by a force of $3\cos(\omega t)$ pounds, find the value of the frequency ω which will cause resonance.

$\frac{1}{4}u'' + 16u = 3\cos(\omega t)$ would be the DE modeling the motion of the body in this case. The associated homogeneous equation $\frac{1}{4}u'' + 16u = 0$ has general solution $u(t) = c_1\cos(8t) + c_2\sin(8t)$ so the natural frequency of the freely oscillating system is 8.

For resonance, we must have the frequency of the forcing term equal to the natural frequency; i.e. $\boxed{\omega = 8}$ is needed for resonance.

4.[16] Find the general solution of $t^2y'' + 3ty' + y = 0$ on the interval $t > 0$.

This is an Euler equation (note the variable coefficients) so $y(t) = t^m$ in $t^2y'' + 3ty' + y = 0$ leads to $m(m-1) + 3m + 1 = 0$. Hence $0 = m^2 + 2m + 1 = (m+1)^2$ so $m = -1$ with multiplicity two. Therefore $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln(t)$ form a fundamental set of solutions on the interval $t > 0$ since

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{-1}\ln(t) \\ -t^{-2} & t^{-1}\cdot t^{-1} - t^{-2}\ln(t) \end{vmatrix} = t^{-3} - t^{-3}\ln(t) + t^{-3}\ln(t) = t^{-3} \neq 0.$$

Thus, $\boxed{y(t) = c_1 t^{-1} + c_2 t^{-1}\ln(t)}$, where c_1 and c_2 are arbitrary constants, is the general solution of the equation on $t > 0$.

5.[17] Use the definition of the Laplace transform, $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$ for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

For which values of s is the Laplace transform of f defined?

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^1 f(t)e^{-st}dt + \int_1^\infty f(t)e^{-st}dt$$

4 pts. to here

$$= \int_0^1 \underbrace{t}_{u}\, \underbrace{e^{-st}dt}_{dv} + \int_1^\infty 1 \cdot e^{-st}dt$$

7 pts to here.

$$= \left.\frac{te^{-st}}{-s}\right|_{t=0}^{1} - \int_0^1 \frac{e^{-st}}{-s}dt + \lim_{M\to\infty}\int_1^M e^{-st}dt$$

10 pts. to here.

$$= \frac{e^{-s}}{-s} - \left.\frac{e^{-st}}{s^2}\right|_{t=0}^{1} + \lim_{M\to\infty}\left.\frac{e^{-st}}{-s}\right|_{t=1}^{M}$$

11 pts. to here

$$= \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \lim_{M\to\infty}\left(\frac{-e^{-sM} + e^{-s}}{s}\right)$$

13 pts. to here. The limit exists only if $s > 0$. In this case, $\lim_{M\to\infty} -e^{-sM} = 0$ so

14 pts. to here.

$$\therefore \mathcal{L}\{f\}(s) = \cancel{\frac{e^{-s}}{-s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \cancel{\frac{e^{-s}}{s}}$$

15 pts. to here

$$= \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$$ provided $\boxed{s > 0}$.

17 pts. to here

6.[17] Use the Laplace transform to solve the initial value problem $y' + y = f(t)$, $y(0) = 2$, where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

Caution: **NO CREDIT** will be awarded for any other method of solution. Note: $f(t) = u_1(t)$.

We take the Laplace transform of both sides of the DE:

$$\mathcal{L}\{y' + y\}(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{u_1(t)\}(s)$$

$$s\mathcal{L}\{y\}(s) - \overset{2}{\cancel{y(0)}} + \mathcal{L}\{y\}(s) = \frac{e^{-s}}{s}.$$

Solving for $\mathcal{L}\{y\}(s)$, we find

$$(s+1)\mathcal{L}\{y\}(s) = 2 + \frac{e^{-s}}{s}$$

$$\mathcal{L}\{y\}(s) = \frac{2}{s+1} + \frac{e^{-s}}{s(s+1)}.$$

Consequently,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{e^{-s}}{s(s+1)}\right\} = 2e^{-t} + \mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s(s+1)}\right\}.$$

Using a partial fraction decomposition, we have

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

where A and B are constants. Then $1 = \left(\frac{A}{s} + \frac{B}{s+1}\right)s(s+1) = A(s+1) + Bs$.

To find A, set $s=0$: $1 = A(1) + B(0)$ so $A = 1$. To find B, set $s = -1$:

$1 = A(0) + B(-1)$ so $B = -1$. Therefore

$$\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s(s+1)}\right\} = f(t-1)u_1(t)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$. Consequently

$$\boxed{y(t) = 2e^{-t} + \left(1 - e^{-(t-1)}\right)u_1(t)}.$$

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. e^{at}	$\frac{1}{s-a}$
3. t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3 \ldots.$
4. $\sin(bt)$	$\frac{b}{s^2 + b^2}$
5. $\cos(bt)$	$\frac{s}{s^2 + b^2}$
6. $f'(t)$	$sF(s) - f(0)$
7. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
8. $e^{ct} f(t)$	$F(s-c)$
9. $u_c(t)$	$\frac{\phantom{e^{-cs}}}{s}$
10. $u_c(t) f(t-c)$	$e^{-cs} F(s)$

2010 Fall Semester, Math 204 Hour Exam II, Master List

100 卌 ||
99 卌 ||
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97 |||
96 卌
95 卌 |||
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168 F's

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Section	Instructor	No. Taking Exam II
A	Wintz	37
B	Wintz	34
C	Wintz	39
D	Willinger	42
E	Heim	44
F	Grow	35
G	Fitch	42
H	Fitch	32
J	He	29
K	Heim	34
L	Singler	33

Number taking exam: 401
Median: 67
Mean: 64.7
Standard Deviation: 22.7

Number receiving A's: 59 — 14.7%
Number receiving B's: 67 — 16.7
Number receiving C's: 60 — 15.0
Number receiving D's: 47 — 11.7
Number receiving F's: 168 — 41.9

$\left.\begin{array}{l}11.7\\41.9\end{array}\right\}$ 53.6%