Mathematics 325 Exam I Summer 2011 Name: Dr. Grow

1.(33 pts.) (a) Solve the partial differential equation $y\sqrt{1-x^2}\,u_x + xu_y = 0$ subject to the condition $u(0,y) = y^2$ for all $-\infty < y < \infty$.

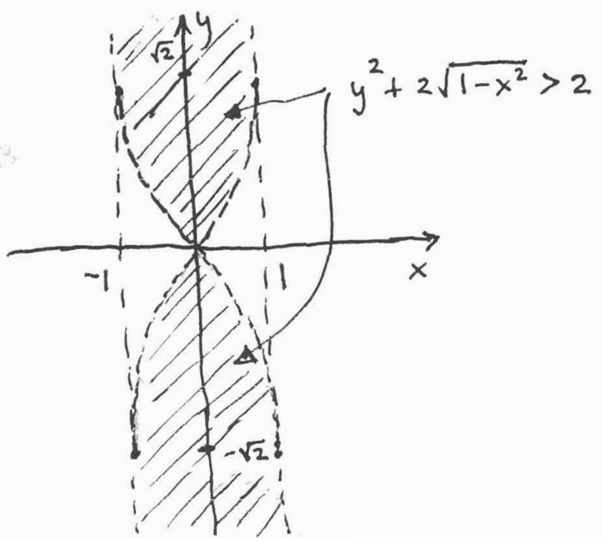
(b) Find and sketch the region in the xy-plane in which the solution in part (a) is uniquely determined.

(a) The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ are $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. For the PDE in (a), $\frac{dy}{dx} = \frac{x}{y\sqrt{1-x^2}}$ so separating variables gives $\int y\,dy = \int \frac{x\,dx}{\sqrt{1-x^2}}$. In the integral in the right member let $w = 1-x^2$. Then $dw = -2x\,dx$ so $\int y\,dy = \int \frac{-\frac{1}{2}dw}{\sqrt{w}}$,

$\frac{y^2}{2} = -\frac{1}{2}\left(\frac{w^{1/2}}{1/2}\right) + c_1 \Rightarrow y^2 + 2\sqrt{1-x^2} = c$. Along such a characteristic curve the solution is constant: $u(x,y) = u\left(x, \pm\sqrt{c - 2\sqrt{1-x^2}}\right) = u\left(0, \pm\sqrt{c-2}\right) = f(c)$.

The general solution of the PDE in (a) is $\boxed{u(x,y) = f\left(y^2 + 2\sqrt{1-x^2}\right)}$ where f is a differentiable function of a single real variable. We need to determine f so the condition is satisfied: $y^2 = u(0,y) = f(y^2+2)$ for all $-\infty < y < \infty$. Let $z = y^2+2$. Then $y^2 = z-2$ so $z-2 = f(z)$ for all $z \geq 2$. Consequently $\boxed{u(x,y) = y^2 + 2\sqrt{1-x^2} - 2}$ is the solution to the problem in (a).

(b) The function f is uniquely determined by $f(z) = z-2$ for all $z \geq 2$. Therefore the solution $u(x,y) = f\left(y^2 + 2\sqrt{1-x^2}\right)$ is uniquely determined for points (x,y) such that $\boxed{y^2 + 2\sqrt{1-x^2} \geq 2}$. This is equivalent to $|y| \geq \sqrt{2 - 2\sqrt{1-x^2}}$. The shaded region in the graph below shows the region of uniqueness for the solution in (a).



2.(34 pts.) (a) Determine the order and type (linear or nonlinear, homogeneous or inhomogeneous, elliptic, parabolic, or hyperbolic) of the partial differential equation

(*) $$u_{xx}-4u_{xy}+4u_{yy}-25u=0.$$

(b) Find the general solution of (*) in the $xy-$plane.

(c) Find the solution of (*) that satisfies the conditions $u(x,0)=e^{3x}$ and $u_y(x,0)=-e^{3x}$ for $-\infty<x<\infty$.

(a) $B^2-4AC=(-4)^2-4(1)(4)=0$. The PDE is $\boxed{\text{second order, linear, homogeneous, and of parabolic type}}$.

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(b) Factoring the differential operator in (*) leads to the equivalent PDE:

7 $\left(\frac{\partial}{\partial x}-2\frac{\partial}{\partial y}\right)^2u-25u=0$. Make the change of variables: $\begin{cases}\xi=\beta x-\alpha y=-2x-y\\ \eta=\alpha x+\beta y=x-2y\end{cases}$

11 or equivalently: $\begin{cases}\xi=2x+y\\ \eta=x-2y\end{cases}$. The chain rule implies that as operators,

$\frac{\partial}{\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}$ and $\frac{\partial}{\partial y}=\frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\xi}-2\frac{\partial}{\partial\eta}$. Therefore

$\frac{\partial}{\partial x}-2\frac{\partial}{\partial y}=2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-2\left(\frac{\partial}{\partial\xi}-2\frac{\partial}{\partial\eta}\right)=5\frac{\partial}{\partial\eta}$ so (*) becomes $\left(5\frac{\partial}{\partial\eta}\right)^2u-25u=0$ or

15 $\frac{\partial^2u}{\partial\eta^2}-u=0$. Then $u=e^{r\eta}$ leads to $r^2e^{r\eta}-e^{r\eta}=0\Rightarrow r^2-1=0\Rightarrow r=\pm1$.

20 Thus $u=c_1(\xi)e^{\eta}+c_2(\xi)e^{-\eta}$ solves the PDE (*); i.e. $\boxed{u(x,y)=f(2x+y)e^{x-2y}+g(2x+y)e^{2y-x}}$

is the general solution of (*) in the xy-plane, with f and g arbitrary C^2-functions of a single real variable.

(c) $u_y=f'(2x+y)e^{x-2y}-2f(2x+y)e^{x-2y}+g'(2x+y)e^{2y-x}+2g(2x+y)e^{2y-x}$. The conditions imply

(1) $-e^{3x}=u_y(x,0)=f'(2x)e^{x}-2f(2x)e^{x}+g'(2x)e^{-x}+2g(2x)e^{-x}$ for all $-\infty<x<\infty$;

(2) $e^{3x}=u(x,0)=f(2x)e^{x}+g(2x)e^{-x}$ for all $-\infty<x<\infty$.

Differentiating (2) gives

(3) $3e^{3x}=2f'(2x)e^{x}+f(2x)e^{x}+2g'(2x)e^{-x}-g(2x)e^{-x}$ for all $-\infty<x<\infty$.

Multiplying (1) by (-2) and adding to (3) yields

$$5e^{3x}=5f(2x)e^{x}-5g(2x)e^{-x}$$

or equivalently

(4) $e^{3x}=f(2x)e^{x}-g(2x)e^{-x}$ for all $-\infty<x<\infty$.

Adding ② and ④ leads to

$$2e^{3x} = 2f(2x)e^{x} \qquad \text{for all } -\infty < x < \infty$$

30 or equivalently $e^{z} = f(z)$ for all $-\infty < z < \infty$. Substituting $e^{2x} = f(2x)$ in ②
gives

$$e^{3x} = e^{2x} \cdot e^{x} + g(x)e^{-x} \qquad \text{for all } -\infty < x < \infty$$

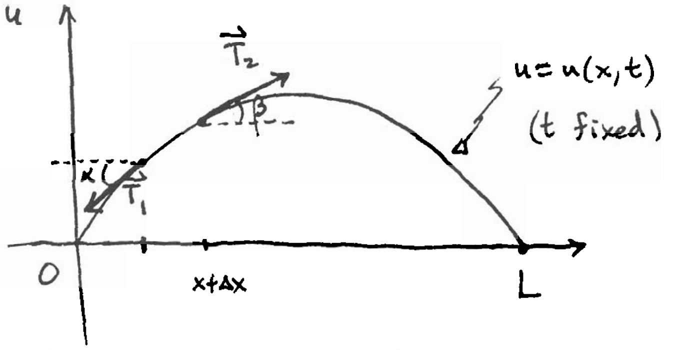
32 or equivalently $0 = g(z)$ for all $-\infty < z < \infty$. Consequently,

$$u(x,y) = f(2x+y)e^{x-2y} + g(2x+y)e^{2y-x} = e^{2x+y} \cdot e^{x-2y} + 0 \cdot e^{2y-x}$$

34 $$\boxed{u(x,y) = e^{3x-y}}$$

solves (*) and satisfies the two auxiliary conditions.

3.(33 pts.) Carefully derive from physical principles the partial differential equation governing the small vibrations of a string in a medium which offers a resistance proportional to velocity.



We apply Newton's second law,

$$\vec{F}_{net} = m\vec{a}$$

to the system of particles comprising the string above the interval $[x, x+\Delta x]$.

Resolving Newton's second law into components yields

(horizontal) ① $|\vec{T}_2|\cos(\beta) - |\vec{T}_1|\cos(\alpha) = 0$

(vertical) ② $\underbrace{|\vec{T}_2|\sin(\beta) - |\vec{T}_1|\sin(\alpha)}_{\text{(net vertical tension force)}} - \underbrace{\int_x^{x+\Delta x} r u_t(\xi,t)\,d\xi}_{\text{(total air resistance force)}} = \underbrace{\int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi}_{\text{(total mass} \times \text{acceleration)}}$

where r is a proportionality constant and $\rho(\xi)$ is the linear mass density at position ξ.

Using $\tan(\alpha) = u_x(x,t)$ and $\tan(\beta) = u_x(x+\Delta x,t)$ we find that $\cos(\alpha) = \dfrac{1}{\sqrt{1+u_x^2(x,t)}}$,

$\sin(\alpha) = \dfrac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$, $\cos(\beta) = \dfrac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}}$, $\sin(\beta) = \dfrac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}$, and from ①,

$|\vec{T}_2| = |\vec{T}_1| \dfrac{\sqrt{1+u_x^2(x+\Delta x,t)}}{\sqrt{1+u_x^2(x,t)}}$. Substitute these expressions in ② and divide by Δx:

$$\frac{|\vec{T}_1|}{\sqrt{1+u_x^2(x,t)}}\left(\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right) - \frac{1}{\Delta x}\int_x^{x+\Delta x} r u_t(\xi,t)\,d\xi = \frac{1}{\Delta x}\int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi .$$

Letting $\Delta x \to 0$ in the above equation produces

$$\frac{|\vec{T}(x,t,u(x,t),u_x(x,t))|}{\sqrt{1+u_x^2(x,t)}}\, u_{xx}(x,t) - r u_t(x,t) = \rho(x) u_{tt}(x,t).$$

For small vibrations, $\sqrt{1+u_x^2(x,t)} \cong 1$ and $|\vec{T}| = \text{constant} = T_0$. Thus

$$\boxed{\rho(x) u_{tt}(x,t) + r u_t(x,t) - T_0 u_{xx}(x,t) = 0}.$$

Math 325

Exam I

Summer 2011

mean: 69.7

standard deviation: 23.1

number taking exam: 24

Distribution of Scores

Range	Grad. Letter Grade	Undergrad. Letter Grade	Frequency
87 – 100	A	A	6
73 – 86	B	B	9
60 – 72	C	B	2
50 – 59	C	C	0
0 – 49	F	D	7