Special Functions

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Chapter 1. Euler, Fourier, Bernoulli, Maclaurin, Stirling

1.1. The Integral Test and Euler's Constant

Suppose we have a series $\sum_{k=1}^{\infty} u_k$ of decreasing terms and a decreasing function f such that $f(k) = u_k$, $k = 1, 2, 3, \ldots$. Also assume f is positive, continuous for $x \geq 1$, and $\lim_{x\to\infty} f(x) = 0$.

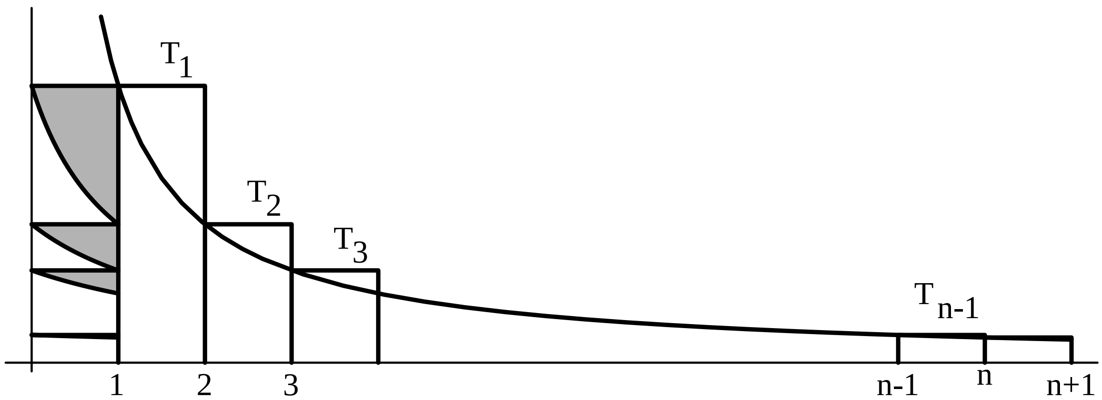


Figure 1

Look at Figure 1 to convince yourself that

$$\sum_{k=1}^{n} u_k = \int_1^n f(x)\,dx + |T_1| + |T_2| + \cdots + |T_{n-1}| + u_n.$$

The left side is the sum of the areas of the rectangles on unit bases with heights $u_1, u_2, \ldots, u_n$ determined from the left end point. $|T_k|$ denotes the area of the triangular-shaped pieces T_k bounded by $x = k+1$, $y = u_k$, and $y = f(x)$. Slide all the T_ks left into the rectangle with opposite vertices $(0,0)$ and $(1, u_1)$ and set

$$A_n = |T_1| + |T_2| + \cdots + |T_{n-1}|$$

Clearly (make sure it *is* clear), $0 < A_2 < A_3 < \cdots < A_n < u_1$, so $\{A_n\}$ is a bounded monotone sequence which has a limit:

$$0 < \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left[|T_1| + |T_2| + \cdots + |T_{n-1}|\right] = C \leq u_1.$$

Let $C_n = A_n + u_n$. We have proved the following result, which should be somewhat familiar.

Theorem 1.1.1 (Integral Test). *Let f be positive, continuous and decreasing on $x \geq 1$. If $f(x) \to 0$ as $x \to \infty$, and if $f(k) = u_k$ for each $k = 1, 2, 3, \ldots$, then the sequence of constants $\{C_n\}_{n=1}^{\infty}$ defined by*

$$\sum_{k=1}^{n} u_k = \int_1^n f(x)\,dx + C_n$$

converges, and $0 \leq \lim_{n\to\infty} C_n = C \leq u_1$.

Corollary 1.1.1 (Calculus Integral Test). *Let f be positive, continuous and decreasing on $x \geq 1$. If $f(x) \to 0$ as $x \to \infty$, and if $f(k) = u_k$ for each $k = 1, 2, 3, \ldots$, then the series*

$$\sum_{k=1}^{\infty} u_k$$

converges if and only if the improper integral

$$\int_1^\infty f(x)\,dx$$

converges.

Example 1.1.1 (The Harmonic Series). *$f(x) = 1/x$, $u_k = 1/k$. By the theorem, the sequence $\{\gamma_n\}$ defined by*

$$\sum_{k=1}^{n} \frac{1}{k} = \int_1^n \frac{1}{x}\,dx + \gamma_n$$

converges, say to γ, where

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right].$$

The number γ is called Euler's constant, or the Euler-Mascheroni constant and has value

$$\gamma = 0.5772\ 15664\ 90153\ 28606\ 06512\ 09008\ldots$$

It is currently not known whether γ is even rational or not, let alone algebraic or transcendental.

Exercise 1.1.1. *Use the above definition and Mathematica or Maple to find the smallest value of n for which γ is correct to four decimal places. Later, we will develop a better way to get accurate approximations of γ.*

Example 1.1.2 (The Riemann Zeta Function). *$f(x) = 1/x^s$, $s > 1$. Now the theorem gives*

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{s-1}\left(1 - \frac{1}{n^{s-1}}\right) + C_n(s)$$

where $0 < C_n(s) < 1$. Let $n \to \infty$, giving

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{s-1} + C(s)$$

with $0 < C(s) < 1$. The summation is the real part of the Riemann zeta function, $\zeta(s)$, a function with many interesting properties, most of which involve its continuation into the complex plane. However, for the real part we get that

$$\zeta(s) = \frac{1}{s-1} + C(s),$$

where $0 < C(s) < 1$.

We shall return to both these examples later.

1.2. Fourier Series

Let $L > 0$ and define the functions $\left\{\phi_k(x)\right\}_{k=1}^{\infty}$ on $[0, L]$ by

$$\phi_k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi x}{L}.$$

Exercise 1.2.1. *Verify that these functions satisfy*

$$\int_0^L \left|\phi_k(x)\right|^2 dx = 1,$$

and, if $j \neq k$,

$$\int_0^L \phi_j(x)\,\phi_k(x)\,dx = 0.$$

If these two conditions are satisfied, we call $\{\phi_k(x)\}_{k=1}^{\infty}$ *an* orthonormal set *over* $[0, L]$.

Now let f be defined on $[0, L]$, and assume that $\int_0^L f(x)\,dx$ and $\int_0^L \left|f(x)\right|^2 dx$ both exist. Define the *Fourier coefficients* of f by

$$a_k = \int_0^L f(x)\,\phi_k(x)\,dx.$$

We want to approximate $f(x)$ by a linear combination of a finite subset of the above orthonormal set.

Exercise 1.2.2. *Show that, for any positive integer* n,

$$\int_0^L \left|f(x) - \sum_{k=1}^{n} c_k\phi_k(x)\right|^2 dx = \int_0^L \left|f(x)\right|^2 dx - \sum_{k=1}^{n}\left|a_k\right|^2 + \sum_{k=1}^{n}\left|c_k - a_k\right|^2,$$

and that the left side of this expression is a minimum when $c_k = a_k$, $k = 1, 2, \ldots, n$. *Note that this is a least squares problem.*

So, $\int_0^L \left|f(x) - \sum_{k=1}^{n} a_k\phi_k(x)\right|^2 dx = \int_0^L \left|f(x)\right|^2 dx - \sum_{k=1}^{n}\left|a_k\right|^2$, and, since the left side cannot be negative,

$$\sum_{k=1}^{n}\left|a_k\right|^2 \le \int_0^L \left|f(x)\right|^2 dx.$$

Since this inequality is true for all n, we have *Bessel's Inequality*:

$$\sum_{k=1}^{\infty}\left|a_k\right|^2 \le \int_0^L \left|f(x)\right|^2 dx.$$

Notice that the important thing about the set $\{\phi_k(x)\}$ was that it was an orthonormal set. The specific sine functions were not the main idea. Given an orthonormal set and a function f, we call $\sum_1^{\infty} a_k\,\phi_k(x)$ the *Fourier series* of f. For our purposes, the most important orthonormal sets are those for which

$$\lim_{n\to\infty} \int_0^L \left|f(x) - \sum_{k=1}^{n} a_k\phi_k(x)\right|^2 dx = 0.$$

Orthonormal sets with this property are *complete*. Some examples of complete orthonormal sets follow. The first two are defined on $[0, L]$ and the third one on $[-L, L]$.

$$\left\{\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\right\}_{k=1}^{\infty} \tag{ON1}$$

$$\left\{\sqrt{\frac{1}{L}}, \sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}, \sqrt{\frac{2}{L}}\cos\frac{2\pi x}{L}, \ldots\right\} \tag{$ON2$}$$

$$\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\cos\frac{\pi x}{L}, \sqrt{\frac{1}{L}}\sin\frac{\pi x}{L}, \sqrt{\frac{1}{L}}\cos\frac{2\pi x}{L}, \sqrt{\frac{1}{L}}\sin\frac{2\pi x}{L}, \ldots\right\} \tag{$ON3$}$$

There are other complete orthonormal sets, some of which we will see later.

For a given orthonormal set, the Fourier series $\sum_{k=1}^{\infty} a_k\,\phi_k(x)$ is equal to $f(x)$ on $-\infty < x < \infty$ for periodic functions f with period $2L$ provided

(1) f is bounded and piecewise monotone on $[-L, L]$,

(2) $\lim_{h\to 0} \dfrac{f(x+h)+f(x-h)}{2} = f(x)$,

(3) f is odd when $(ON1)$ is the orthonormal set,

(4) f is even when $(ON2)$ is the orthonormal set.

1.3. Bernoulli Functions and Numbers

The *Bernoulli functions*, $B_0(x), B_1(x), B_2(x), \ldots$, satisfy the following conditions on $-\infty < x < \infty$:

$$B_0(x) = 1$$

$$B_n'(x) = B_{n-1}(x),\ n = 1, 2, 3, \ldots *$$

$$\int_0^1 B_n(x)\,dx = 0,\ n = 1, 2, 3, \ldots$$

$$B_n(x+1) = B_n(x),\ n = 1, 2, 3, \ldots$$

Exercise 1.3.1. *Show that there exist constants $B_0, B_1, B_2, \ldots$ such that for $0 < x < 1$*

$$B_0(x) = \frac{B_0}{0!0!}$$

$$B_1(x) = \frac{B_0 x}{0!1!} + \frac{B_1}{1!0!}$$

$$B_2(x) = \frac{B_0 x^2}{0!2!} + \frac{B_1 x}{1!1!} + \frac{B_2}{2!0!}$$

$$B_3(x) = \frac{B_0 x^3}{0!3!} + \frac{B_1 x^2}{1!2!} + \frac{B_2 x}{2!1!} + \frac{B_3}{3!0!}$$

etc.

Exercise 1.3.2. *Show that , when $n \geq 2$, $B_n = n!\,B_n(0)$*

Exercise 1.3.3. *Show that on $(0, 1)$,*

$$0!B_0(x) = B_0$$

* Except when $n = 1$ or 2 and x is an integer.

$$\begin{aligned}1!B_1(x) &= B_0x + B_1\\ 2!B_2(x) &= B_0x^2 + 2B_1x + B_2\\ 3!B_3(x) &= B_0x^3 + 3B_1x^2 + 3B_2x + B_3\\ &\mathit{etc.}\end{aligned}$$

Some authors define the Bernoulli polynomials (on $(-\infty, \infty)$) to be the right hand sides of the above equations. If, in the future, you encounter Bernoulli functions or polynomials, be sure to check what is intended by a particular author.

Exercise 1.3.4. *Show that for $n \geq 2$, $B_n(1) = B_n(0)$.*

Exercise 1.3.5. *Compute B_n for $n = 0, 1, 2, 3, \ldots, 12$.*

Exercise 1.3.6. *Show that $B_1(x) = x - \lfloor x \rfloor - 1/2$ for $-\infty < x < \infty$ and x not an integer. [Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.]*

Since $B_1(x) = x - \frac{1}{2}$ on $(0, 1)$ and is an odd function on $(-1, 1)$ (do you see why?) we can expand it in Fourier series using $(ON1)$ with $L = 1$. The Fourier coefficients are

$$a_k = \sqrt{2}\int_0^1 (x - \frac{1}{2})\sin(k\pi x)\,dx = -\frac{\sqrt{2}}{k\pi}\left(\frac{1 + (-1)^k}{2}\right).$$

Thus, $a_k = 0$ if k is odd, and $a_k = -\frac{\sqrt{2}}{k\pi}$ if k is even. This gives

$$B_1(x) = -2\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{2k\pi} = -\frac{2}{2\pi}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k}.$$

Integrate term by term and use the fact that $B_2'(x) = B_1(x)$ to get

$$B_2(x) = \frac{2}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^2}.$$

Similarly,

$$B_{2n+1}(x) = (-1)^{n+1}\frac{2}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k^{2n+1}},$$

and

$$B_{2n}(x) = (-1)^{n+1}\frac{2}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2n}}.$$

Exercise 1.3.7. *The work above with the Fourier series was done formally, without worrying about whether the results were meaningful. Prove that the formulas for $B_2(x)$, $B_{2n+1}(x)$, and $B_{2n}(x)$ are correct by showing that the series converge and satisfy the properties of the Bernoulli functions.*

Exercise 1.3.8. *Use Mathematica or Maple to plot graphs of $B_1(x)$, $B_2(x)$, and $B_3(x)$ on $0 \leq x \leq 4$. Also graph the Fourier approximations of $B_1(x)$, $B_2(x)$, and $B_3(x)$ using $n = 2$, $n = 5$, and $n = 50$.*

Example 1.3.1 (Some Values of the Riemann Zeta Function). *Since $B_n(0) = B_n/n!$, we have $B_2(0) = 1/12$. Therefore,*

$$\frac{1}{12} = \frac{2}{(2\pi)^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

and so we get

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{(2\pi)^2}{(12)(2)} = \frac{\pi^2}{6}.$$

Exercise 1.3.9. *Find* $\zeta(4)$, $\zeta(6)$, *and* $\zeta(8)$.

Van der Pol used to say that those who know these formulas are mathematicians and those who do not are not.

1.4. The Euler-Maclaurin Formulas

Let p and q be integers and assume f is differentiable (as many times as needed) for $p \le x \le q$. Let k be an integer, $p \le k < q$. Then

$$\int_k^{k+1} f(x)\,dx = \int_k^{k+1} f(x)B_0(x)\,dx = \lim_{\epsilon\to 0}\int_{k+\epsilon}^{k+1-\epsilon} f(x)B_1'(x)\,dx.$$

Integration by parts gives

$$\int_k^{k+1} f(x)\,dx = \lim_{\epsilon\to 0}\left[f(x)B_1(x)\Big]_{k+\epsilon}^{k+1-\epsilon} - \int_{k+\epsilon}^{k+1-\epsilon} f'(x)B_1(x)\,dx\right] = \frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f'(x)B_1(x)\,dx.$$

Adding between p and q, we get

$$\int_p^q f(x)\,dx = \sum_{k=p}^{q-1}\int_k^{k+1} f(x)\,dx = \sum_{k=p}^{q} f(k) - \frac{f(p)+f(q)}{2} - \int_p^q f'(x)B_1(x)\,dx.$$

A slight rearrangement produces the first *Euler-Maclaurin Formula*:

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \int_p^q f'(x)B_1(x)\,dx. \qquad (EM1)$$

This is a useful formula for estimating sums.

Additional Euler-Maclaurin formulas can be obtained by further integration by parts.

Exercise 1.4.1. *Derive the following: (Remember that* $B_j = 0$ *if* $j \ge 3$ *and odd.)*

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \frac{f'(q)-f'(p)}{12} - \int_p^q f''(x)B_2(x)\,dx. \qquad (EM2)$$

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \frac{f'(q)-f'(p)}{12} + \int_p^q f'''(x)B_3(x)\,dx. \qquad (EM3)$$

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \sum_{j=2}^{m}\left(f^{(j-1)}(q) - f^{(j-1)}(p)\right)\frac{B_j}{j!} + (-1)^{m+1}\int_p^q f^{(m)}(x)B_m(x)\,dx. \qquad (EMm)$$

Example 1.4.1. *In (EM3), let $f(x) = x^2$, $p = 0$, and $q = n$. Since $f^m(x) = 0$ for $m \geq 3$ we get*

$$\begin{aligned}\sum_{k=0}^{n} k^2 &= \int_0^n x^2\,dx + \frac{0+n^2}{2} + \frac{2n-0}{12}\\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

This is much neater than mathematical induction.

Example 1.4.2. *In (EMm), let $p = 0$, $q = n$, $m = s$, and $f(x) = x^s$, where s is a positive integer other than 1. Then*

$$\begin{aligned}\sum_{k=1}^{n} k^s &= \frac{n^{s+1}}{s+1} + \frac{n^s}{2} + \sum_{j=2}^{s} \frac{f^{(j-1)}(n)B_j}{j!} + (-1)^{s+1}\int_0^n s!B_s(x)\,dx\\ &= \frac{n^{s+1}}{s+1} + \frac{n^s}{2} + \sum_{j=2}^{s} \frac{s(s-1)\dots(s-j+2)n^{s-j+1}B_j}{j!}\\ &= n^s + \frac{1}{s+1}\sum_{j=0}^{s} \binom{s+1}{j} n^{s+1-j}B_j\end{aligned}$$

Exercise 1.4.2. *Fill in the details in the last example and get formulas for $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^4$.*

In some cases, as $x \to \infty$, $f^{(m)}(x) \to 0$ for m large enough. When the integral in the following expression converges, we can define a constant C_p by

$$C_p = \frac{f(p)}{2} - \sum_{j=2}^{m} \frac{f^{(j-1)}(p)B_j}{j!} + (-1)^{m+1}\int_p^\infty f^{(m)}(x)B_m(x)\,dx.$$

Exercise 1.4.3. *Show that C_p is independent of m by showing that the right side is unchanged when m is replaced by $m+1$. Integration by parts helps.*

Subtract the C_p equation from (EMm) to get

$$\sum_{k=p}^{q} f(k) = C_p + \int_p^q f(x)\,dx + \frac{f(q)}{2} + \sum_{j=2}^{m} \frac{f^{(j-1)}(q)B_j}{j!} + (-1)^m\int_q^\infty f^{(m)}(x)B_m(x)\,dx.$$

We solve for C_p to get

$$C_p = \sum_{k=p}^{q} f(k) - \int_p^q f(x)\,dx - \frac{f(q)}{2} - \sum_{j=2}^{m} \frac{f^{(j-1)}(q)B_j}{j!} - (-1)^m\int_q^\infty f^{(m)}(x)B_m(x)\,dx.$$

Example 1.4.3 (Euler's Constant). *Let $f(x) = 1/x$, $p = 1$, $q = n$, and (at first) $m = 3$. Then the penultimate formula involving C_p, now C_1, gives*

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &= C_1 + \int_1^n \frac{1}{x}\,dx + \frac{1}{2n} + \sum_{j=2}^{3} \frac{f^{(j-1)}(n)B_j}{j!} + (-1)^3\int_n^\infty f^{(3)}(x)B_3(x)\,dx\\ &= \log n + C_1 + \frac{1}{2n} - \frac{B_2}{n^2 2!} - \int_n^\infty -6x^{-4}B_3(x)\,dx\\ &= \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + 6\int_n^\infty \frac{B_3(x)}{x^4}\,dx.\end{aligned}$$

Exercise 1.4.4. *Fill in the details to this point in the example, especially why γ can replace C_1. Then, assuming γ is known, obtain bounds on the last integral and approximate $\sum_{k=1}^{n} \frac{1}{k}$ for $n = 10, 50,$ and 100. How close are your estimates?*

(Continuation of Example 1.4.3). *If now $q = 10$, and m is arbitrary, the last formula for C_1 $(= \gamma)$ gives*

$$\begin{aligned}
\gamma &= \sum_{k=1}^{10} \frac{1}{k} - \log 10 - \frac{1}{20} - \sum_{j=2}^{m} \frac{(-1)^{j-1}(j-1)!B_j}{10^j j!} - (-1)^m \int_{10}^{\infty} \frac{(-1)^m m! B_m(x)}{x^{m+1}}\,dx \\
&= \sum_{k=1}^{10} \frac{1}{k} - \log 10 - \frac{1}{20} + \sum_{j=2}^{m} \frac{B_j}{10^j j} - \int_{10}^{\infty} \frac{m! B_m(x)}{x^{m+1}}\,dx.
\end{aligned}$$

Exercise 1.4.5. *Prove that if $m = 10$ in the last formula for γ, then the integral is less than 10^{-12}, and so the other terms can be used to compute γ correct to at least ten decimal places. Do this computation. To best appreciate the formula, do the computation by hand, assuming that you know $\log 10$ to a sufficient number of places (you have already found exact values for the Bernoulli numbers you need).* ($\log 10 =$ 2.3025 85092 994...)

1.5 The Stirling Formulas

This section is a (long) derivation of the Stirling formulas for $\log(z!)$ and $z!$. As you work through the section, think about how the steps fit together.

Exercise 1.5.1. *Let $p = 1$, $q = n$, $m \geq 2$, and $f(x) = \log(z + x)$ for $z > -1$. Use (EMm) to get*

$$\begin{aligned}
\sum_{k=1}^{n} \log(z+k) = (z + n + \frac{1}{2}) \log(z+n) &- (z + \frac{1}{2}) \log(z+1) - n + 1 \\
&+ \sum_{j=2}^{m} \frac{B_j}{j(j-1)} \left(\frac{1}{(z+n)^{j-1}} - \frac{1}{(z+1)^{j-1}} \right) \qquad (1.5.1) \\
&+ \int_{1}^{n} \frac{(m-1)! B_m(x)}{(z+x)^m}\,dx.
\end{aligned}$$

Put $z = 0$ in (1.5.1) to get

$$\log(n!) = (n + \frac{1}{2}) \log n - n + 1 + \sum_{j=2}^{m} \frac{B_j}{j(j-1)} \left(\frac{1}{n^{j-1}} - 1 \right) + \int_{1}^{n} \frac{(m-1)! B_m(x)}{x^m}\,dx. \qquad (1.5.2)$$

In the next chapter we will see how *Wallis' formulas*, (see also A&S, 6.1.49)

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2n} x\,dx &= \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2}, \\
\int_0^{\pi/2} \sin^{2n+1} x\,dx &= \frac{2^{2n}(n!)^2}{(2n)!} \frac{1}{2n+1},
\end{aligned}$$

can be used to prove that

$$\lim_{n\to\infty} \frac{(2n)!\sqrt{n\pi}}{2^{2n}(n!)^2} = 1. \qquad (1.5.3)$$

Accept this result for now - you will have a chance to prove it later! From (1.5.3) we get

$$\lim_{n\to\infty} \left[\log\left((2n)!\right) + \log\sqrt{n\pi} - 2n\log 2 - 2\log\left(n!\right)\right] = 0. \tag{1.5.4}$$

Substitute for $\log((2n)!)$ and $\log(n!)$ in (1.5.4) using (1.5.2) and simplify to get

$$\begin{aligned}\lim_{n\to\infty} \Bigg[&\frac{1}{2}\log 2 - 1 + \frac{1}{2}\log\pi \\ &+ \sum_{j=2}^{m} \frac{B_j}{j(j-1)} \left(\frac{1}{(2n)^{j-1}} - \frac{2}{n^{j-1}} + 1 \right) \\ &+ \int_1^{2n} \frac{(m-1)!B_m(x)}{x^m}\,dx - 2\int_1^{n} \frac{(m-1)!B_m(x)}{x^m}\,dx \Bigg] = 0.\end{aligned}$$

More simplification yields

$$\log\sqrt{2\pi} - 1 + \sum_{j=2}^{m} \frac{B_j}{j(j-1)} - \int_1^{\infty} \frac{(m-1)!B_m(x)}{x^m}\,dx = 0 \tag{1.5.5}$$

Exercise 1.5.2. *Show that*

$$\int_1^{n} \frac{(m-1)!B_m(x)}{x^m}\,dx - \int_1^{\infty} \frac{(m-1)!B_m(x)}{x^m}\,dx = -\int_0^{\infty} \frac{(m-1)!B_m(x)}{(n+x)^m}\,dx \tag{1.5.6}$$

Add (1.5.5) to (1.5.2), and use (1.5.6) to get

$$\log\left(n!\right) = \log\sqrt{2\pi} + (n+\frac{1}{2})\log n - n + \sum_{j=2}^{m} \frac{B_j}{j(j-1)n^{j-1}} - \int_0^{\infty} \frac{(m-1)!B_m(x)}{(n+x)^m}\,dx. \tag{1.5.7}$$

Clearly, for integers $z > 0$,

$$\begin{aligned} z! &= \lim_{n\to\infty} 1\cdot 2\cdot 3\cdot \ldots \cdot z \\ &= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot z(z+1)(\ldots(z+n)}{(z+1)\ldots(z+n)} \\ &= \lim_{n\to\infty} \left[\left(\frac{n!\,n^z}{(z+1)\ldots(z+n)} \right) \left(\frac{n+1}{n} \right) \left(\frac{n+2}{n} \right) \cdots \left(\frac{n+z}{n} \right) \right]. \end{aligned}$$

Since each of the last factors has limit one, we have (see A&S, 6.1.2), for $z > -1$,

$$z! = \lim_{n\to\infty} \frac{n!\,n^z}{(z+1)(z+2)\ldots(z+n)}. \tag{1.5.8}$$

Taking logs,

$$\log\left(z!\right) = \lim_{n\to\infty} \left[\log\left(n!\right) + z\log n - \sum_{k=1}^{n} \log\left(z+k\right) \right]. \tag{1.5.9}$$

Substitute from (1.5.7) and (1.5.1) to get

$$\begin{aligned}\log\left(z!\right) = \log\sqrt{2\pi} &+ (z+\frac{1}{2})\log\left(z+1\right) - (z+1) \\ &+ \sum_{j=2}^{m} \frac{B_j}{j(j-1)(z+1)^{j-1}} - \int_1^{\infty} \frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned} \tag{1.5.10}$$

Exercise 1.5.3. *Show that*

$$\lim_{n\to\infty}\left[\left(z+n+\frac{1}{2}\right)(\log(z+n)-\log n)\right]=z \tag{1.5.11}$$

and use this fact to get (1.5.10).

If $z>0$, add $\log(z+1)$ to both sides of (1.5.10)

$$\begin{aligned}\log((z+1)!) &= \log\sqrt{2\pi}+(z+\frac{3}{2})\log(z+1)-(z+1)\\ &\quad+\sum_{j=2}^{m}\frac{B_j}{j(j-1)(z+1)^{j-1}}-\int_1^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned}$$

Finally, replace $z+1$ by z:

$$\begin{aligned}\log(z!) &= \log\sqrt{2\pi}+(z+\frac{1}{2})\log z-z\\ &\quad+\sum_{j=2}^{m}\frac{B_j}{j(j-1)z^{j-1}}-\int_0^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned} \tag{1.5.12}$$

Note that for $z=n$, (1.5.12) is identical to (1.5.7).

For $z\in\mathbb{C}-\{z\mid\Re(z)\le 0\}$, everything on the right side of (1.5.12) is analytic. Analytic continuation then makes (1.5.12) valid for all complex z not on the non-positive real axis. To make the notation more compact, let

$$E(z)=\sum_{j=2}^{m}\frac{B_j}{j(j-1)z^{j-1}}-\int_0^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx, \tag{1.5.13}$$

so that (1.5.12) becomes

$$\log(z!)=\log\sqrt{2\pi}+(z+\frac{1}{2})\log z-z+E(z), \tag{1.5.14}$$

or, equivalently,

$$z!=\sqrt{2\pi z}\;z^z e^{-z}e^{E(z)}. \tag{1.5.15}$$

Equations (1.5.14) and (1.5.15) are the *Stirling formulas* for $\log(z!)$ and $z!$. Equation (1.5.15) can be thought of as *defining* $z!$ when z is not a positive integer. See A&S, 6.1.37 and 6.1.38. The term $E(z)$ is small and can be bounded by simple functions, so the Stirling formulas can be used to estimate $z!$ and $\log(z!)$ quite accurately.

Exercise 1.5.4. *For z real and positive, show that*

$$0<E(z)<\frac{1}{12z},$$

and

$$\frac{1}{12z}-\frac{1}{360z^3}<E(z)<\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}.$$

Exercise 1.5.5. *Use the Stirling formulas to estimate* $5!$ *and* $\log(5!)$ *within 3 decimal places, then do the same for* $5.5!$ *and* $\log(5.5!)$. *Think about how you could find these values without a fancy calculator or computer.*

Chapter 2. The Gamma Function

2.1. Definition and Basic Properties

Although we will be most interested in real arguments for the gamma function, the definition is valid for complex arguments. See Chapter 6 in A&S for more about the gamma function.

Definition 2.1.1. *For z a complex number with $\Re(z) > 0$, $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}\,dt$.*

Theorem 2.1.1 (Difference Equation). $\Gamma(z+1) = z\,\Gamma(z)$.

Proof. For the proof we apply integration by parts to the integral in the definition of $\Gamma(z)$.

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt = \frac{t^z e^{-t}}{z}\Big]_0^\infty + \int_0^\infty \frac{t^z e^{-t}}{z}\,dt.$$

Thus, $z\,\Gamma(z) = \int_0^\infty e^{-t}t^z\,dt = \Gamma(z+1)$. ♠

Theorem 2.1.2 (Factorial Equivalence). $\Gamma(n+1) = n!$ *for* $n = 0, 1, 2, \ldots$.

Proof. By direct calculation in the definition, $\Gamma(1) = 1$. Repeated use of Theorem 2.1.1 gives $\Gamma(n+1) = n!\,\Gamma(1) = n!$. ♠

Theorem 2.1.3. *If x is real and positive, then* $\displaystyle\lim_{x\to 0+}\Gamma(x) = +\infty$.

Proof.

$$\Gamma(x) > \int_0^1 e^{-t}t^{x-1}\,dt > \frac{1}{e}\int_0^1 t^{x-1}\,dt.$$

The last integral is an improper integral, so

$$\int_0^1 t^{x-1}\,dt = \lim_{\epsilon\to 0+}\int_\epsilon^1 t^{x-1}\,dt = \lim_{\epsilon\to 0+}\left[\frac{1}{x} - \frac{\epsilon^x}{x}\right] = \frac{1}{x}.$$

So, $\Gamma(x) > \dfrac{1}{e\,x}$ for $x > 0$, and thus $\Gamma(x) \to \infty$ as $x \to 0$. ♠

The gamma function is often referred to as the "continuous version of the factorial", or words to that effect. If we are going to say this, we need to prove that $\Gamma(x)$ is continuous. The next theorem uses the Weierstrass M-test for improper integrals, something you should be familiar with for series. The result works similarly for integrals. (Find your advanced calculus book and review the Weierstrass M-test if necessary.)

Theorem 2.1.4 (Continuity of Γ). *The gamma function is continuous for all real positive x.*

Proof. Assume $x_0 > 0$ and choose a and b such that $0 < a < x_0 < b$. Then the integral $\int_1^\infty e^{-t}t^{x-1}\,dt$ converges uniformly on $[a, b]$ by the Weierstrass M-test because $\left|e^{-t}t^{x-1}\right| < e^{-t}t^{b-1}$ and $\int_1^\infty e^{-t}t^{b-1}\,dt$ converges.

The integral $\int_0^1 e^{-t}t^{x-1}\,dt$ is proper for $x \in [a, b]$ if $a \geq 1$. If $0 < a < 1$, then this integral also converges uniformly by the Weierstrass M-test since $\left|e^{-t}t^{x-1}\right| < t^{a-1}$ and $\int_0^1 t^{a-1}\,dt$ converges.

Combining these results, we see that the integral defining $\Gamma(x)$ converges uniformly on $[a, b]$, and the integrand is continuous in x and t. By an advanced calculus theorem, this makes Γ continuous on $[a, b]$ and thus continuous at x_0. ♠

Exercise 2.1.1. *Show that, for x real and positive,* $\lim_{x \to 0+} x\,\Gamma(x) = 1$.

The domain of $\Gamma(x)$ can be extended to include values between consecutive negative integers. For $n = 1, 2, 3, \ldots$, and $-n < x < -n+1$, define $\Gamma(x)$ by

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)}.$$

In this way, $\Gamma(x)$ is defined for all $x \neq 0, -1, -2, \ldots$.

Exercise 2.1.2. *Show that* $\Gamma(x+1) = x\,\Gamma(x)$ *for all* $x \neq 0, -1, -2, \ldots$.

We know that $\Gamma(x)$ becomes infinite as $x \to 0+$ and as $x \to \infty$, but what happens in between? Differentiating, we get, for $0 < x < \infty$,

$$\Gamma'(x) = \frac{d}{dx}\int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^\infty e^{-t}t^{x-1}\log t\,dt,$$

$$\Gamma''(x) = \int_0^\infty e^{-t}t^{x-1}(\log t)^2\,dt.$$

Since the integrand in $\Gamma''(x)$ is positive for $0 < x < \infty$, so is $\Gamma''(x)$. Thus, the graph of $\Gamma(x)$ is concave up on $(0, \infty)$.

The technique used in the proof of the following theorem is one everyone should know.

Theorem 2.1.5. $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Proof. Consider the following regions in the first quadrant of the plane, shown in Figure 2.

$$S = \{(x, y)\,|\,0 \le x \le R,\ 0 \le y \le R\}$$

$$D_1 = \{(x, y)\,|\,x \ge 0,\ y \ge 0,\ x^2 + y^2 \le R^2\}$$

$$D_2 = \{(x, y)\,|\,x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 2R^2\}$$

Clearly,

$$\iint_{D_1} e^{-x^2-y^2}dA < \iint_S e^{-x^2-y^2}dA < \iint_{D_2} e^{-x^2-y^2}dA.$$

Use polar coordinates on the outside integrals and rectangular coordinates on the middle one to get

$$\int_0^{\pi/2}\int_0^R r\,e^{-r^2}dr\,d\theta < \int_0^R\int_0^R e^{-x^2}e^{-y^2}dx\,dy < \int_0^{\pi/2}\int_0^{\sqrt{2}R} r\,e^{-r^2}dr\,d\theta,$$

$$\int_0^{\pi/2}\frac{1-e^{-R^2}}{2}\,d\theta < \left(\int_0^R e^{-x^2}dx\right)\left(\int_0^R e^{-y^2}dy\right) < \int_0^{\pi/2}\frac{1-e^{-2R^2}}{2}\,d\theta,$$

$$\frac{\pi}{4}(1-e^{-R^2}) < \left(\int_0^R e^{-x^2}dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2}).$$

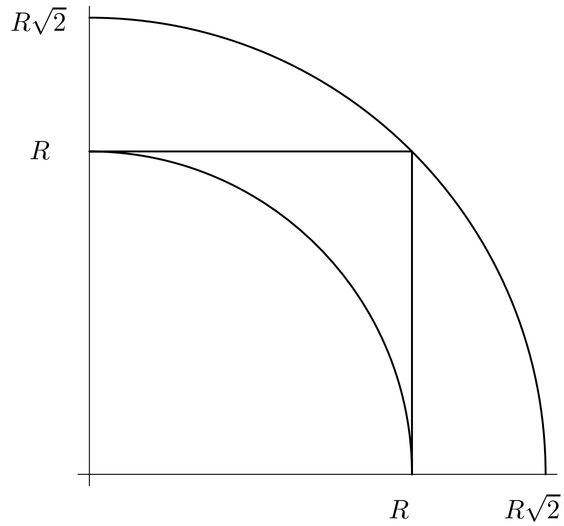


Figure 2.1.1

Taking limits as $R \to \infty$ yields

$$\left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4},$$

and so $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. ♠

Corollary 2.1.5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt$. Let $t = y^2$ to get

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-y^2} y^{-1} 2y\, dy = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}. \ ♠$$

Exercise 2.1.3. *Prove* $\Gamma(n + \frac{1}{2}) = \dfrac{(2n)!\sqrt{\pi}}{2^{2n} n!}$.

Exercise 2.1.4. *For* $0 < x < \infty$, *prove* $\Gamma(x) = 2\displaystyle\int_0^\infty e^{-t^2} t^{2x-1} dt$.

Exercise 2.1.5. *Show that* $f(x) = \displaystyle\int_0^\infty e^{-t^2} \cos(xt)\, dt = \frac{\sqrt{\pi}}{2} e^{-x^2/4}$. *[Hint: Find and solve a differential equation satisfied by* f*.]*

Exercise 2.1.6. *Find all positive numbers* T *such that* $\int_0^T x^{-\log x}\, dx = \int_T^\infty x^{-\log x}\, dx$, *and evaluate the integrals.*

2.2. The Beta Function, Wallis' Product

Another special function defined by an improper integral and related to the gamma function is the *beta function*, denoted $B(x, y)$.

Definition 2.2.1. $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, *for* $x > 0, \quad y > 0$.

If both $x > 1$ and $y > 1$, then the beta function is given by a proper integral and convergence is not a question. However, if $0 < x < 1$ or $0 < y < 1$, then the integral is improper. Convince yourself that in these cases the integral converges, making the beta function well-defined. We now develop some of the properties of $B(x,y)$. Unless otherwise stated, we assume x and y are in the first quadrant.

Theorem 2.2.1 (Symmetry). $B(x,y) = B(y,x)$.

Proof. In the definition, make the change of variable $u = 1 - t$. ♠

Theorem 2.2.2. $B(x,y) = 2\displaystyle\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}dt$.

Proof. Make the change of variable $t = \sin^2 u$. ♠

Theorem 2.2.3. $B(x,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt$.

Proof. Let $t = \frac{u}{1+u}$. ♠

Exercise 2.2.1. *Fill in the details in the proofs of Theorems 2.2.1 - 2.2.3.*

Theorem 2.2.4 (Relation to the Gamma Function). $B(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$.

Proof. The proof uses the method employed in the proof of Theorem 2.1.5. (When a trick is used twice it becomes a method!) From Exercise 2.1.4, we know $\Gamma(x) = 2\displaystyle\int_0^\infty e^{-t^2}t^{2x-1}dt$, so consider the function given by $G(t,u) = t^{2x-1}u^{2y-1}e^{-t^2-u^2}$. Integrate G (with respect to t and u) over the three regions shown in Figure 2, using polar coordinates in the quarter-circles as before. The inequalities become

$$\begin{aligned}
&\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, d\theta \int_0^R r^{2x+2y-1}e^{-r^2}dr \\
&\qquad < \int_0^R t^{2x-1}e^{-t^2}dt \int_0^R u^{2y-1}e^{-u^2}du \\
&\qquad\qquad < \int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, d\theta \int_0^{\sqrt{2}R} r^{2x+2y-1}e^{-r^2}dr.
\end{aligned}$$

As $R \to \infty$, we see from Exercise 2.1.4 and Theorem 2.2.2 that the center term approaches $\Gamma(x)\,\Gamma(y)/4$, and the outside terms approach $B(x,y)\,\Gamma(x+y)/4$. Thus, $B(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$. ♠

Exercise 2.2.2 (Dirichlet Integrals 1). *Show that*

$$\iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}dV = \frac{\Gamma(\frac{\alpha}{2})\,\Gamma(\frac{\beta}{2})\,\Gamma(\frac{\gamma}{2})}{8\,\Gamma(\frac{\alpha+\beta+\gamma}{2}+1)},$$

where V is the region in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$. [Let $x^2 = u$, $y^2 = v$, and $z^2 = w$ to transform the region of integration into a tetrahedron. After another substitution later, recognize the beta function integral so you can use Theorem 2.2.4.]

Exercise 2.2.3 (Dirichlet Integrals 2). *Show that*

$$\iiint_V x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dV = \frac{a^\alpha b^\beta c^\gamma}{pqr} \frac{\Gamma(\frac{\alpha}{p})\,\Gamma(\frac{\beta}{q})\,\Gamma(\frac{\gamma}{r})}{\Gamma(1 + \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r})},$$

where V is the region in the first octant bounded by the coordinate planes and $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$.

Exercise 2.2.4. *Prove Wallis' Formulas:*

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2} = \frac{\sqrt{\pi}\,\Gamma(n + \frac{1}{2})}{2(n!)},$$
$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2^{2n}(n!)^2}{(2n)!} \frac{1}{2n+1} = \frac{\sqrt{\pi}\, n!}{2\,\Gamma(n + \frac{3}{2})}.$$

[Use Exercise 2.1.3, Theorem 2.2.2, and Theorem 2.2.4.]

An interesting fact about Wallis' formulas is that $\frac{(2n)!}{2^{2n}(n!)^2}$ is the probability of getting exactly n heads when $2n$ coins are tossed.

Exercise 2.2.5. *An excellent aproximation to the probability of getting exactly n heads when $2n$ coins are tossed is given by $\frac{1}{\sqrt{n\,\pi}}$. Use Mathematica or Maple to convince yourself that this is true. (The proof will come later.)*

Theorem 2.2.5 (Wallis' Product). $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \cdots.$

Proof. Let P_n be the partial product of the first n factors on the right side. We must show that $\lim_{n\to\infty} P_n = \frac{\pi}{2}$. From Exercise 2.2.4,

$$\frac{\int_0^{\pi/2} \sin^{2n} x\, dx}{\int_0^{\pi/2} \sin^{2n+1} x\, dx} = \frac{\Gamma(n + \frac{1}{2})}{n!} \cdot \frac{\Gamma(n + \frac{3}{2})}{n!}.$$

By Theorem 2.1.1 and some algebra,

$$\frac{\Gamma(n + \frac{1}{2})}{n!} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2(n-1)} \cdots \frac{3}{2 \cdot 2} \cdot \frac{1}{2 \cdot 1} \sqrt{\pi},$$
$$\frac{\Gamma(n + \frac{3}{2})}{n!} = \frac{2n+1}{2n} \cdot \frac{2n-1}{2(n-1)} \cdots \frac{3}{2 \cdot 1} \cdot \frac{1}{2} \sqrt{\pi}.$$

So the quotient above is

$$\frac{\int_0^{\pi/2} \sin^{2n} x\, dx}{\int_0^{\pi/2} \sin^{2n+1} x\, dx} = \frac{1}{P_{2n}} \cdot \frac{\pi}{2}.$$

Again using Exercise 2.2.4, we have

$$\frac{\int_0^{\pi/2} \sin^{2n+1} x\, dx}{\int_0^{\pi/2} \sin^{2n-1} x\, dx} = \frac{2n}{2n+1},$$

or

$$\int_0^{\pi/2} \sin^{2n-1} x\,dx = \frac{2n+1}{2n} \int_0^{\pi/2} \sin^{2n+1} x\,dx.$$

Since $\sin x$ is increasing and $0 \le \sin x \le 1$ on $[0, \pi/2]$,

$$0 < \int_0^{\pi/2} \sin^{2n+1} x\,dx < \int_0^{\pi/2} \sin^{2n} x\,dx < \int_0^{\pi/2} \sin^{2n-1} x\,dx.$$

Divide by $\int_0^{\pi/2} \sin^{2n+1} x\,dx$ to get

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} < \frac{2n+1}{2n}.$$

Clearly, as $n \to \infty$, the middle term $\to 1$, giving us

$$\lim_{n\to\infty} \frac{1}{P_{2n}} \cdot \frac{\pi}{2} = 1 \text{ and so } \lim_{n\to\infty} P_{2n} = \frac{\pi}{2}.$$

Since $\displaystyle\lim_{n\to\infty} P_{2n+1} = \lim_{n\to\infty} \frac{2n+2}{2n+1} P_{2n} = \frac{\pi}{2}$, the proof is complete. ♠

Corollary 2.2.5. $\displaystyle\lim_{n\to\infty} \frac{(2n)!\sqrt{n\pi}}{2^{2n}(n!)^2} = 1.$

Exercise 2.2.6 (Progress as Promised). *Prove Corollary 2.2.5.*

Exercise 2.2.7. *Prove that the approximation in Exercise 2.2.5 is correct by showing that*

$$\frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{1 - \frac{1-\theta_n}{2n+1}} \cdot \frac{1}{\sqrt{n\pi}}$$

for some θ_n satisfying $0 < \theta_n < 1$. [The θ_n comes from using the Mean Value Theorem on one of the inequalities in the proof of Theorem 2.2.5.]

2.3. The Reflection Formula

First, a Fourier series warm-up.

Exercise 2.3.1. *Expand $f(x) = |x|$ for $-\pi \le x \le \pi$, and $f(x+2\pi) = f(x)$ in Fourier series. Use this result to show that*

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \quad \textit{and} \quad \frac{\pi^2}{24} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.$$

Now that you have Fourier series back at the top level in your mind, the next exercise will be needed soon.

Exercise 2.3.2. *Expand $f(x) = \cos(zx)$ for $-\pi \le x \le \pi$, and $f(x+2\pi) = f(x)$ in Fourier series (treat z as a constant) to get*

$$\cos(zx) = \frac{2z}{\pi} \sin(\pi z) \left[\frac{1}{2z^2} + \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{z^2 - k^2} \right].$$

The following theorem is stated in terms of complex z, but no arguments in the proof require complex analysis, so feel free to think of the z as a real number.

Theorem 2.3.1 (The Reflection Formula). *$\Gamma(z)\Gamma(1-z) = \pi\ \csc(\pi\, z)$ for $0 < \Re(z) < 1$.*

Proof. Let $x = 0$ in the series of Exercise 2.3.2 to get

$$\pi\ \csc(\pi\, z) = 2z\left[\frac{1}{2z^2} - \frac{1}{z^2 - 1^2} + \frac{1}{z^2 - 2^2} - \cdots\right].$$

For $0 < \Re(z) < 1$, Theorems 2.2.3 and 2.2.4 give

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) = B(z, 1-z) &= \int_0^\infty \frac{x^{z-1}}{1+x}dx \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_1^\infty \frac{x^{z-1}}{1+x}dx \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_1^0 \frac{-t^{-z}}{1+t}dt \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_0^1 \frac{x^{-z}}{1+x}dx \\
&= \int_0^1 x^{z-1}dx + \int_0^1 \frac{x^{-z} - x^z}{1+x}dx \\
&= \frac{1}{z} + \int_0^1 (x^{-z} - x^z)(1 - x + x^2 - x^3 + \cdots)\, dx \\
&= \frac{1}{z} + \int_0^1 \sum_{k=0}^{\infty}(x^{-z+k} - x^{z+k})\, dx \\
&= \frac{1}{z} - \left(\frac{1}{1+z} - \frac{1}{1-z}\right) + \left(\frac{1}{2+z} - \frac{1}{2-z}\right) - \cdots \\
&= 2z\left[\frac{1}{2z^2} - \frac{1}{z^2 - 1^2} + \frac{1}{z^2 - 2^2} - \cdots\right].
\end{aligned}$$

The proof is complete provided we can justify the term-by-term integration. Denote by $S_n(x)$ and $R_n(x)$ the n^{th} partial sum and remainder of the series $\sum_{k=0}^{\infty}(-1)^k x^k (x^{-z} - x^z)$. We need to show that $\int_0^1 R_n(x)\, dx \to 0$ as $n \to \infty$.

$$\int_0^1 |R_n(x)|\, dx = \int_0^1 \frac{x^{n+1}(x^{-z} - x^z)}{1+x}dx = \int_0^1 x^n \left[\frac{x^{1-z} - x^{1+z}}{1+x}\right] dx.$$

Since $0 < z < 1$, the function $\frac{x^{1-z} - x^{1+z}}{1+x}$ is continuous in x on $[0, 1]$, and so there is a number M, such that $\dfrac{x^{1-z} - x^{1+z}}{1+x} \le M$ for all $z \in (0, 1)$. Thus,

$$\int_0^1 |R_n(x)|\, dx \le \int_0^1 M x^n\, dx = \frac{M}{n+1}.$$

This completes the proof. ♠

Example 2.3.1 (Another Route to Wallis' Product). *Let $x = \pi$ in the series of Exercise 2.3.2 to get*

$$\cos(\pi\, z) = \frac{2z\ \sin(\pi\, z)}{\pi}\left[\frac{1}{2z^2} + \frac{1}{z^2 - 1^2} + \frac{1}{z^2 - 2^2} + \frac{1}{z^2 - 3^2} + \cdots\right],$$

$$\pi \cot(\pi z) - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Integrate both sides with respect to z *from* 0 *to* x, $-1 < x < 1$. *(This time term-by-term integration is valid because the series converges uniformly for* $|z| < 1$.*)*

$$\begin{aligned} \int_0^x \left(\frac{\pi \cos(\pi z)}{\sin(\pi z)} - \frac{1}{z} \right) dz &= \sum_{k=1}^{\infty} \log|z^2 - k^2| \Bigg]_0^x \\ \log(\sin(\pi x)) - \log x - \lim_{z \to 0} [\log(\sin(\pi z)) - \log z] &= \sum_{k=1}^{\infty} \log\left(\frac{k^2 - x^2}{k^2}\right) \\ \log\left(\frac{\sin(\pi x)}{\pi x}\right) &= \sum_{k=1}^{\infty} \log\left(1 - \frac{x^2}{k^2}\right). \end{aligned}$$

This is equivalent to

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \quad \textit{for } -1 < x < 1.$$

Let $x = \frac{1}{2}$ *and factor the term in the product to get*

$$\begin{aligned} 1 &= \frac{\pi}{2} \left[\left(\frac{1}{2} \cdot \frac{3}{2}\right) \left(\frac{3}{4} \cdot \frac{5}{4}\right) \left(\frac{5}{6} \cdot \frac{7}{6}\right) \cdots \right] \quad \textit{or} \\ \frac{\pi}{2} &= \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \cdots \end{aligned}$$

In most of the following exercises use Mathematica or Maple to do graphs and numerical work. Proofs are, of course, still your responsibility. If you use Mathematica or Maple to do an integral whose value is given in terms of a special function, think of the result as a theorem to prove.

Exercise 2.3.3. *Evaluate* $\int_0^{\infty} e^{-st} \sqrt{t}\, dt$, *which gives the Laplace transform of* $\sqrt{t}$.

Exercise 2.3.4. *Evaluate* $\int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{x-1} dt$ *and* $\int_0^1 (\log t)^{x-1}\, dt$. *When is the second one real-valued?*

Exercise 2.3.5. *Plot the graph of* $y = 1/\Gamma(x)$ *for* $-4 \le x \le 10$. *Using the computer, find the first 8 terms in the Taylor series expansion of* $1/\Gamma(x)$ *around* $x = 0$. *Do the first 8 terms give a good approximation of the value of* $\Gamma(5)$? *How about* $\Gamma(2)$? *Compare with the values from Stirling's formula, and revise, if necessary, your opinion of old Stirling.*

Exercise 2.3.6. *Show that* $B(x,x) = 2^{1-2x} B(x, \frac{1}{2})$ *for* $0 < x < \infty$. *Plot the graph of* $y = B(x,x)$ *and* $y = 2^{1-2x} B(x, \frac{1}{2})$ *on* $(0, 10]$.

Exercise 2.3.7. *Show that* $\sqrt{\pi}\, \Gamma(2x) = 2^{2x-1} \Gamma(x)\, \Gamma(x + \frac{1}{2})$, *for* $0 < x < \infty$. *[Exercise 2.3.6 should help.]*

Exercise 2.3.8. *Evaluate* $f(t) = \int_0^{\pi/2} (\sin(2x))^{2t-1} dx$, *and plot the graph of* f *on* $(0, 10]$.

Exercise 2.3.9. *Plot the graph of $x^{2/3} + y^{2/3} = 1$, and find the area inside the curve. [Parameterize the curve in terms of trig functions.]*

2.4. Stirling and Weierstrass

It is a good idea to have a feeling for the order of magnitude of $n!$ and $\Gamma(x)$, especially in comparison with other things that "get real big real fast". The following theorem, due to Stirling, addresses this topic.

Theorem 2.4.1 (Stirling). $\lim\limits_{n\to\infty} \dfrac{\left(\frac{n}{e}\right)^n \sqrt{2\pi\, n}}{n!} = 1.$

Proof. Let $a_n = \dfrac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}}$. We will show that $a_n \to \sqrt{2\pi}$ as $n \to \infty$. In Corollary 2.2.5, we can write

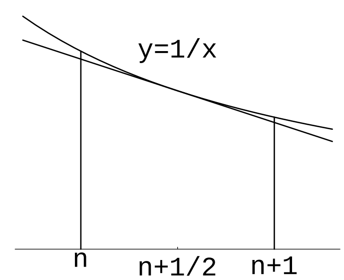
$$\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} = \frac{a_n^2}{\sqrt{2}a_{2n}} = \frac{(n!)^2 \left(\frac{2n}{e}\right)^{2n} \sqrt{2n}}{\sqrt{2}\left(\frac{n}{e}\right)^{2n} n(2n)!}.$$

Assuming $\lim\limits_{n\to\infty} a_n = r \neq 0$, Corollary 2.2.5 gives

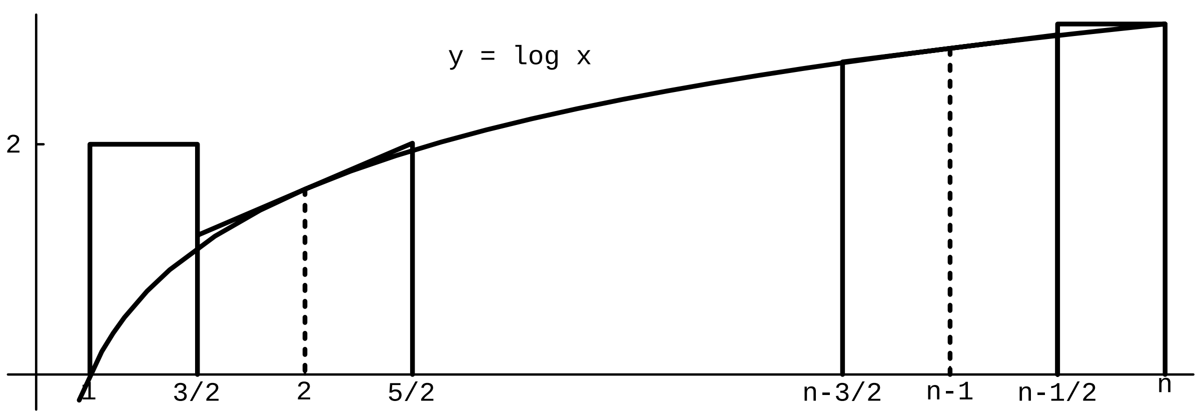
$$\sqrt{\pi} = \frac{r^2}{r\sqrt{2}} \text{ or } r = \sqrt{2\pi}.$$

The proof will be complete when we (you!) show that the sequence $\{a_n\}$ has a nonzero limit. This is done in an exercise. ♠

Exercise 2.4.1. *Show* $\log\left(1 + \dfrac{1}{n}\right) > \dfrac{2}{2n+1}$ *for* $n = 1, 2, 3, , \ldots.$ *[Hint: See figure.]*



Exercise 2.4.2. *For the sequence $\{a_n\}$ in Theorem 2.4.1 prove that $\{a_n\}$ is a bounded monotonic sequence, and thus has a limit. Further, show that this limit is ≥ 1. [Hint: See figure below.]*



We now prepare for Weierstrass' infinite product representation of the gamma function, which involves Euler's constant, γ.

Lemma 2.4.1. *For* $0 \le x \le 1$, $\displaystyle\lim_{n\to\infty} \frac{\Gamma(x+n)}{\Gamma(n)\ n^x} = 1$.

Proof. Since $x \ge 0$ and $x-1 \le 0$, if $0 \le t \le n$, we get $t^x \le n^x$ and $n^{x-1} \le t^{x-1}$ so that

$$n^{x-1}\int_0^n e^{-t}t^n dt \le \int_0^n e^{-t}t^{n+x-1}dt \le n^x \int_0^n e^{-t}t^{n-1}dt. \tag{2.4.1}$$

Similarly, if $n \le t \le \infty$, we have $n^x \le t^x$ and $t^{x-1} \le n^{x-1}$ so that

$$n^x \int_n^\infty e^{-t}t^{n-1}dt \le \int_n^\infty e^{-t}t^{n+x-1}dt \le n^{x-1}\int_n^\infty e^{-t}t^n dt. \tag{2.4.2}$$

In (2.4.2) integrate the outside integrals by parts to get

$$-e^{-n}n^{n+x-1} + n^{x-1}\int_n^\infty e^{-t}t^n dt \le \int_n^\infty e^{-t}t^{n+x-1}dt \le e^{-n}n^{n+x-1} + n^x \int_n^\infty e^{-t}t^{n-1}dt. \tag{2.4.3}$$

Add (2.4.1) and (2.4.3) and note the appearance of gamma functions to get

$$-e^{-n}n^{n+x-1} + n^{x-1}\Gamma(n+1) \le \Gamma(x+n) \le e^{-n}n^{n+x-1} + n^x\Gamma(n).$$

Divide by $n^x\Gamma(n)$ and simplify to get

$$-\frac{e^{-n}n^n}{n!} + 1 \le \frac{\Gamma(x+n)}{\Gamma(n)\ n^x} \le \frac{e^{-n}n^n}{n!} + 1.$$

By Theorem 2.4.1 (Stirling) $\displaystyle\lim_{n\to\infty}\frac{e^{-n}n^n}{n!} = 0$, which completes the proof. ♠

Lemma 2.4.2. *For* $0 \le x \le \infty$, $\displaystyle\lim_{n\to\infty} \frac{\Gamma(x+n)}{\Gamma(n)\ n^x} = 1$.

Sketch of Proof. Use induction and the fact that

$$\frac{\Gamma(x+n)}{\Gamma(n)\ n^x} = \frac{x-1+n}{n}\cdot\frac{\Gamma(x-1+n)}{\Gamma(n)\ n^{x-1}}. \ ♠$$

The following result of Weierstrass can be, and sometimes is, used to *define* the gamma function instead of the integral in Definition 2.1.1. See, for example, *A Course of Modern Analysis* by Whittaker and Watson.

Theorem 2.4.2 (Weierstrass). *If* $x > 0$ *and* γ *denotes Euler's constant, then*

$$\frac{1}{\Gamma(x)} = x\,e^{\gamma x}\prod_{k=1}^\infty \left(1+\frac{x}{k}\right)e^{-x/k}.$$

Proof. Let

$$P_n = x\prod_{k=1}^{n-1}\left(1+\frac{x}{k}\right)e^{-x/k} = \left(x\prod_{k=1}^{n-1}(x+k)\right)\left(\prod_{k=1}^{n-1}\frac{1}{k}\right)\left(\prod_{k=1}^{n-1}e^{-x/k}\right) = \left(\prod_{k=0}^{n-1}(x+k)\right)\frac{1}{\Gamma(n)}e^{-xH_{n-1}},$$

where $H_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$. By Theorem 2.1.1,

$$\begin{aligned}\frac{1}{\Gamma(x)} &= \frac{\prod_{k=0}^{n-1}(x+k)}{\Gamma(x+n)} = \frac{P_n\Gamma(n)e^{xH_{n-1}}}{\Gamma(x+n)}\cdot\frac{n^x}{n^x} \\ &= \frac{\Gamma(n)\ n^x}{\Gamma(x+n)}P_n e^{xH_{n-1}}e^{-x\log n} = \frac{\Gamma(n)\ n^x}{\Gamma(x+n)}P_n e^{(H_{n-1}-\log n)x}.\end{aligned}$$

By the definition of γ and Lemmas 2.4.1 and 2.4.2, we get $\displaystyle\lim_{n\to\infty} P_n = \frac{e^{-\gamma x}}{\Gamma(x)}$. ♠

Weierstrass' theorem connects the gamma function and Euler's constant. This connection can be further exploited.

Theorem 2.4.3. $\Gamma'(1) = -\gamma$.

Proof. The logarithmic derivative of the gamma function, i.e., the derivative of $\log(\Gamma(x))$, is called the *digamma function* and is denoted by $\psi(x)$. Taking logs and differentiating in Theorem 2.4.2 gives

$$-\psi(x) = \gamma + \frac{1}{x} - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right),$$

and, for $x = 1$,

$$-\psi(1) = \gamma + 1 - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \gamma.$$

Since $\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)}$ and $\Gamma(1) = 1$, the proof is complete. ♠

Exercise 2.4.3. *Show that* $\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}$ *for* $n \geq 2$ *and an integer. Find* $\Gamma'(2)$, $\Gamma'(3)$, *and* $\Gamma'(17)$.

Exercise 2.4.4. *Show that* $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ *and* $\psi(n+\frac{1}{2}) = -\gamma - 2\log 2 + 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right)$ *for* $n \geq 1$ *and an integer.*

Exercise 2.4.5 (Difference Equation). *Show that* $\psi(x+1) = \psi(x) + \frac{1}{x}$.

2.5. Evaluation of a Class of Infinite Products

Suppose u_n is a rational function of n written as

$$u_n = \frac{A(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_j)}.$$

In order for the product $\prod_{n=1}^{\infty} u_n$ to converge absolutely, we need $A = 1$ and $j = k$, because otherwise $u_n \not\to 1$ as $n \to \infty$. Thus, we are led to the product

$$P = \prod_{n=1}^{\infty} u_n = \prod_{n=1}^{\infty}\frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)}$$

where the general term u_n can be written as

$$\begin{aligned} u_n &= \left(1-\frac{a_1}{n}\right)\cdots\left(1-\frac{a_k}{n}\right)\left(1-\frac{b_1}{n}\right)^{-1}\cdots\left(1-\frac{b_k}{n}\right)^{-1} \\ &= 1 - \frac{a_1+a_2+\cdots+a_k-b_1-b_2-\cdots-b_k}{n} + O(n^{-2}) \end{aligned}$$

where the Binomial Theorem was used to expand the negative powers. Absolute convergence forces the $\frac{1}{n}$ term to be 0, or $a_1 + \cdots + a_k - b_1 - \cdots - b_k = 0$. Thus, $\exp\left(\frac{a_1+a_2+\cdots+a_k-b_1-b_2-\cdots-b_k}{n}\right) = 1$, and can multiply u_n without changing P.

$$P = \prod_{n=1}^{\infty}\frac{\left(1-\frac{a_1}{n}\right)e^{\frac{a_1}{n}}\left(1-\frac{a_2}{n}\right)e^{\frac{a_2}{n}}\cdots\left(1-\frac{a_k}{n}\right)e^{\frac{a_k}{n}}}{\left(1-\frac{b_1}{n}\right)e^{\frac{b_1}{n}}\left(1-\frac{b_2}{n}\right)e^{\frac{b_2}{n}}\cdots\left(1-\frac{b_k}{n}\right)e^{\frac{b_k}{n}}}$$

Now use Theorem 2.4.2 to express P as

$$P=\prod_{n=1}^{\infty}\frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)}=\prod_{i=1}^{k}\frac{\Gamma(1-b_i)}{\Gamma(1-a_i)}. \tag{2.5.1}$$

Exercise 2.5.1. *Fill in all the details in the derivation of equation (2.5.1).*

Exercise 2.5.2. *Evaluate, if possible,* $\prod_{n=1}^{\infty}\frac{(n+2)(n+5)(n+7)}{(n+4)^2(n+6)}$.

Example 2.5.1. *Evaluate* $x\left(1-\frac{x}{1^n}\right)\left(1-\frac{x}{2^n}\right)\left(1-\frac{x}{3^n}\right)\cdots$, *where n is a positive integer.*

$$P=x\prod_{k=1}^{\infty}\left(1-\frac{x}{k^n}\right)=x\prod_{k=1}^{\infty}\left(\frac{k^n-(x^{1/n})^n}{k^n}\right).$$

Let $\alpha=e^{\frac{2\pi i}{n}}$ so that $\alpha^n=e^{2\pi i}=1$. Note that the n n^{th} roots of 1 are $\alpha^0,\alpha^1,\ldots,\alpha^{n-1}$. Let $z=x^{1/n}$. Then $k^n-z^n=(k-\alpha^0z)(k-\alpha^1z)\cdots(k-\alpha^{n-1}z)$. (Remember that the "unknown" is k, the product index.) We can now write

$$P=z^n\prod_{k=1}^{\infty}\frac{(k-\alpha^0z)(k-\alpha^1z)\cdots(k-\alpha^{n-1}z)}{(k-0)(k-0)\cdots(k-0)}.$$

Clearly, $b_1+b_2+\cdots+b_n=0$. The sum $a_1+a_2+\cdots+a_n$ is the same as $z(\alpha^0+\alpha^1+\cdots+\alpha^{n-1})$, and since the α^j's are the roots of a polynomial of degree n with no degree $(n-1)$ term, their sum is 0. Hence the product is absolutely convergent, and equation (2.5.1) may be applied to get

$$P=z^n\prod_{j=0}^{n-1}\frac{\Gamma(1)}{\Gamma(1-\alpha^jz)}=z^n\frac{1}{-\alpha^0z\Gamma(-\alpha^0z)(-\alpha^1z)\Gamma(-\alpha^1z)\cdots(-\alpha^{n-1}z)\Gamma(-\alpha^{n-1}z)}$$
$$=\frac{1}{(-1)^n\alpha^{1+2+\cdots+(n-1)}\Gamma(-x^{1/n})\Gamma(-\alpha^1x^{1/n})\cdots\Gamma(-\alpha^{n-1}x^{1/n}}$$

Since $\alpha^{1+2+\cdots+(n-1)}=(-1)^{n-1}$, we get

$$P=\frac{1}{-\Gamma(-x^{1/n})\Gamma(-\alpha^1x^{1/n})\cdots\Gamma(-\alpha^{n-1}x^{1/n})}.$$

Exercise 2.5.3. *Evaluate* $(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\cdots$.

Exercise 2.5.4. *Evaluate* $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$, $\prod_{n=k+1}^{\infty}\left(1-\frac{k^2}{n^2}\right)$, *and* $\prod_{n=k+1}^{\infty}\left(1-\frac{k^m}{n^m}\right)$. *In the last product, m is a positive integer.*

Chapter 3. Elliptic Integrals and Elliptic Functions

3.1. Motivational Examples

Example 3.1.1 (The Pendulum). *A simple, undamped pendulum of length L has motion governed by the differential equation*

$$u'' + \frac{g}{L}\sin u = 0, \tag{3.1.1}$$

where u is the angle between the pendulum and a vertical line, g is the gravitational constant, and $'$ is differentiation with respect to time. Consider the "energy" function (some of you may recognize this as a Lyapunov function):

$$E(u, u') = \frac{(u')^2}{2} + \int_0^u \frac{g}{L}\sin z\, dz = \frac{(u')^2}{2} + \frac{g}{L}(1 - \cos u).$$

For $u(0)$ and $u'(0)$ sufficiently small (3.1.1) has a periodic solution. Suppose $u(0) = A$ and $u'(0) = 0$. At this point the energy is $\frac{g}{L}(1 - \cos A)$, and by conservation of energy, we have

$$\frac{(u')^2}{2} + \frac{g}{L}(1 - \cos u) = \frac{g}{L}(1 - \cos A).$$

Simplifying, and noting that at first u is decreasing, we get

$$\frac{du}{dt} = -\sqrt{\frac{2g}{L}}\sqrt{\cos u - \cos A}.$$

This DE is separable, and integrating from $u = 0$ to $u = A$ will give one-fourth of the period. Denoting the period by T, and realizing that the period depends on A, we get

$$T(A) = 2\sqrt{\frac{2L}{g}}\int_0^A \frac{du}{\sqrt{\cos u - \cos A}}. \tag{3.1.2}$$

If $A = 0$ or $A = \pi$ there is no motion (do you see why this is so physically?), so assume $0 < A < \pi$. Let $k = \sin\frac{A}{2}$, making $\cos A = 1 - 2k^2$, and also let $\sin\frac{u}{2} = k\sin\theta$. Substituting into (3.1.2) gives

$$T(A) = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.3}$$

Since $0 < A < \pi$, we have $0 < k < 1$, and the integral in (3.1.3) cannot be evaluated in terms of elementary functions. This integral is the complete elliptic integral of the first kind *and is denoted by K, $K(k)$, or $K(m)$ (where $m = k^2$).*

$$K = K(k) = K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.4}$$

A slight generalization, replacing the upper limit by a variable ϕ, with $0 \le \phi \le \pi/2$, yields the incomplete elliptic integral of the first kind, *denoted $K(\phi\,|\,k)$ or $K(\phi\,|\,m)$.*

$$K(\phi\,|\,k) = K(\phi\,|\,m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.5}$$

Exercise 3.1.1. *Fill in all the details in Example 3.1.1.*

Example 3.1.2 (Circumference of an Ellipse). *Let an ellipse be given by* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$*, where we assume that* $a < b$*. In parametric form, the ellipse is* $x = a\cos\theta$*,* $y = b\sin\theta$*, and the circumference* L *is given by*

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta \\ &= 4b\int_0^{\frac{\pi}{2}} \sqrt{1 - (1 - \frac{a^2}{b^2})\sin^2\theta}\, d\theta \\ &= 4b\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, d\theta, \end{aligned}$$

where $k^2 = 1 - \frac{a^2}{b^2}$*. Again, the integral cannot be evaluated in terms of elementary functions except in degenerate cases (*$k = 0$ *or* $k = 1$*). Integrals of the form*

$$E = E(k) = E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, d\theta \tag{3.1.6}$$

are called complete elliptic integrals of the second kind (as before, $m = k^2$*), and integrals of the form*

$$E(\phi \,|\, k) = E(\phi \,|\, m) = \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\, d\theta \tag{3.1.7}$$

are called incomplete elliptic integrals of the second kind.

Exercise 3.1.2. *Show that, in the setting of Example 3.1.2,* k *is the eccentricity of the ellipse.*

There are elliptic integrals of the third kind, denoted by Π. As before, if the upper limit in the integral is $\pi/2$, the integral is called complete.

$$\Pi(\phi \,|\, k, N) = \int_0^{\phi} \frac{d\theta}{(1 + N\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.8}$$

Unfortunately, I don't know any nice motivating examples for this case. The following is lifted verbatim from Whittaker and Watson, page 523, and and assumes knowledge of things we have not covered. Understanding of it is something to which you are encouraged to aspire.

Example 3.1.3 (Rigid Body Motion). *It is evident from the expression of* $\Pi(u, a)$ *in terms of Theta-functions that if* u*,* a*,* k *are real, the average rate of increase of* $\Pi(u, a)$ *as* u *increases is* $Z(a)$*, since* $\Theta(u \pm a)$ *is periodic with respect to the real period* $2K$*. This result determines the mean precession about the invariable line in the motion of a rigid body relative to its centre [Whittaker and Watson were British] of gravity under forces whose resultant passes through its centre of gravity. It is evident that, for purposes of computation, a result of this nature is preferable to the corresponding result in terms of Sigma-functions and Weierstrassian Zeta-Functions, for the reasons that the Theta-functions have a specially simple behaviour with respect to their real period - the period which is of importance in Applied Mathematics - and that the q-series are much better adapted for computation than the product by which the Sigma-function is most simply defined.*

Before we consider elliptic integrals in general, look back at Example 3.1.1. By either the Binomial Theorem or Taylor's Theorem,

$$\left(1 - k^2\sin^2\theta\right)^{-1/2} = 1 + \frac{1}{2}\,k^2\sin^2\theta + \frac{1}{2}\cdot\frac{3}{4}\,k^4\sin^4\theta + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,k^6\sin^6\theta + \cdots.$$

and so

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \frac{(2n)!}{2^{2n}(n!)^2}\, k^{2n} \sin^{2n}\theta\, d\theta.$$

By Wallis' formula, we get

$$K = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right].$$

This series can be used to approximate the value of K. Note that $|k| < 1$ is necessary for convergence.

Exercise 3.1.3. *Express E as a power series in powers of k.*

The functions K, E, and Π are tabulated in A&S and are part of Mathematica and Maple.

3.2. General Definition of Elliptic Integrals

If $R(x,y)$ is a rational algebraic function of x and y, the integral $\int R(x,y)\,dx$ can be evaluated in terms of elementary functions if $y = \sqrt{ax+b}$ or $y = \sqrt{ax^2+bx+c}$. Things are not so nice if y^2 is a cubic or quartic, however.

Exercise 3.2.1. *Evaluate $\int xy\,dx$ and $\int \frac{1}{y}\,dx$ when $y = \sqrt{ax+b}$ and $y = \sqrt{ax^2+bx+c}$.*

Definition 3.2.1. *If $R(x,y)$ is a rational function of x and y and y^2 is a cubic or quartic polynomial in x with no repeated factors, then the integral $\int R(x,y)\,dx$ is an elliptic integral.*

So, the trigonometry in the above examples notwithstanding, elliptic integrals are concerned with integrating algebraic functions that you couldn't handle in second-semester calculus. Given an elliptic integral, the problem is to reduce it to a recognizable form.

Example 3.2.1. *Evaluate $I = \int_1^{\infty} \frac{dx}{\sqrt{x^4-1}}$. Here, $y^2 = x^4 - 1$ and $R(x,y) = \frac{1}{y}$. A sequence of substitutions will convert the integral to a form we have seen. First, let $x = \frac{1}{t}$ to get $I = \int_0^1 \frac{dt}{\sqrt{1-t^4}}$. Next, let $t = \sin\phi$, giving $I = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{2-\cos^2\phi}}$. Finally, let $\phi = \frac{\pi}{2} - \theta$ to get*

$$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2-\sin^2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{\sqrt{2}} K\left(\tfrac{1}{2}\right).$$

Exercise 3.2.2. *Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}}$ and $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$ in terms of complete elliptic integrals, and use the tables in A&S to get numerical values. [Hint: Although not the only way, the substitution $\cos(\cdot) = \cos^2 u$ can be used at some stage in both problems.] Express these integrals in terms of the gamma function using Theorems 2.2.2 and 2.2.4. Also try integrating these directly using Mathematica or Maple.*

3.3. Evaluation of Elliptic Integrals

A systematic way to evaluate elliptic integrals is desirable. Since $R(x, y)$ is a rational function of x and y, we can write $R(x,y) = \dfrac{y\,P(x,y)\,Q(x,-y)}{y\,Q(x,y)\,Q(x,-y)}$, where P and Q are polynomials. Now $Q(x,y)Q(x,-y)$ is an even function of y, making it a polynomial in x and y^2, and thus a polynomial in x. When $y\,P(x,y)\,Q(x,y)$ is expanded, even powers of y can be written as polynomials in x and odd powers of y can be written as (polynomials in x) times y, so the numerator of R is linear in y. We then have

$$R(x,y) = \frac{R_1(x) + y\,R_2(x)}{y} = R_2(x) + \frac{R_1(x)}{y}$$

where R_1 and R_2 are rational functions of x. The integration problem has been reduced to

$$\int R(x,y)\,dx = \int R_2(x)\,dx + \int \frac{R_1(x)}{y}\,dx.$$

The first integral can be done by second-semester calculus methods, and the second one will be studied further. Recall that y^2 is a cubic or quartic in x. Think of a cubic as a quartic with the coefficient of x^4 equal to 0. Then the following factorization is useful.

Theorem 3.3.1. *Any quartic in x with no repeated factors can be written in the form*

$$\left[A_1(x-\alpha)^2 + B_1(x-\beta)^2\right]\left[A_2(x-\alpha)^2 + B_2(x-\beta)^2\right]$$

where, if the coefficients in the quartic are real, then the constants $A_1, B_1, A_2, B_2, \alpha$, and β are real.

Proof. Any quartic $Q(x)$ with real coefficients can be expressed as $Q(x) = S_1(x)\,S_2(x)$ where S_1 and S_2 are quadratics. The complex roots (if any) of the quartic occur in conjugate pairs, so there are three cases.

> **Case 1. Four real roots.** Call the roots $\{r_i\}$, and assume $r_1 < r_2 < r_3 < r_4$. Let $S_1(x) = (x-r_1)(x-r_2)$ and $S_2(x) = (x-r_3)(x-r_4)$, with an appropriate constant multiplier in case the coefficient of x^4 is not 1. Note that the roots of S_1 and S_2 do not interlace.
>
> **Case 2. Two real roots and two complex roots.** Denote the real roots by r_1 and r_2, and the complex roots by $\rho_1 \pm \rho_2 i$. Let $S_1(x) = (x-r_1)(x-r_2)$ and $S_2(x) = x^2 - 2\rho_1\,x + (\rho_1^2 + \rho_2^2)$.
>
> **Case 3. Four complex roots.** Call the roots $\rho_1 \pm \rho_2 i$ and $\rho_3 \pm \rho_4 i$. Let $S_1(x) = x^2 - 2\rho_1\,x + (\rho_1^2 + \rho_2^2)$ and $S_2(x) = x^2 - 2\rho_3\,x + (\rho_3^2 + \rho_4^2)$.

In case y^2 is a cubic, we simply eliminate one real factor in Case 1 or Case 2. Case 3 will not apply if y^2 is a cubic. So, in general, we have

$$S_1(x) = a_1x^2 + 2b_1x + c_1 \text{ and } S_2(x) = a_2x^2 + 2b_2x + c_2.$$

Now we look for constants λ such that $S_1(x) - \lambda S_2(x)$ is a perfect square. Since $S_1(x) - \lambda S_2(x)$ is simply a quadratic in x, it is a perfect square if and only if the discriminant is zero, or $(a_1-\lambda a_2)(c_1-\lambda c_2)-(b_1-\lambda b_2)^2 = 0$. This discriminant is a quadratic in λ, and has two roots, λ_1 and λ_2. We get

$$S_1(x) - \lambda_1 S_2(x) = (a_1 - \lambda_1 a_2)\left[x + \frac{b_1 - \lambda_1 b_2}{a_1 - \lambda_1 b_2}\right]^2 = (a_1 - \lambda_1 a_2)(x-\alpha)^2, \tag{3.3.1}$$

$$S_1(x) - \lambda_2 S_2(x) = (a_1 - \lambda_2 a_2)\left[x + \frac{b_1 - \lambda_2 b_2}{a_1 - \lambda_2 b_2}\right]^2 = (a_1 - \lambda_2 a_2)(x-\beta)^2, \tag{3.3.2}$$

Now solve equations (3.3.1) and (3.3.2) for $S_1(x)$ and $S_2(x)$ to get the required forms. ♠

Example 3.3.1. *Consider* $Q(x) = 3x^4 - 16x^3 + 24x^2 - 16x + 4 = (3x^2 - 4x + 2)(x^2 - 4x + 2)$. *Mathematica or Maple is handy for the factorization. For* $S_1(x) - \lambda S_2(x)$ *to be a perfect square, we need* $(3 - \lambda)(2 - 2\lambda) - (-2 + 2\lambda)^2 = 0$, *which has solutions* $\lambda_1 = -1$ *and* $\lambda_2 = 1$. *So we get* $S_1(x) = 2(x - 1)^2 + (x - 0)^2$ *and* $S_2(x) = 2(x - 1)^2 - (x - 0)^2$.

Exercise 3.3.1. *Fill in all the details in the proof of Theorem 3.3.1 and in Example 3.3.1.*

If $b_1 = b_2 = 0$, there appears to be a breakdown in the form specified in Theorem 3.3.1. In this case you can set up S_1 and S_2 with complex coefficients to get the form in the theorem. In practice, this will not be necessary, however, because the form of the integral will already be one toward which you are working.

In the integral $\int \frac{R_1(x)}{y}\, dx$, with the denominator written as in Theorem 3.3.1, make the substitution

$$t = \frac{x - \alpha}{x - \beta}\ ,\ dx = (x - \beta)^2(\alpha - \beta)^{-1}\, dt.$$

This gives

$$y^2 = \left[A_1(x - \alpha)^2 + B_1(x - \beta)^2\right]\left[A_2(x - \alpha)^2 + B_2(x - \beta)^2\right] = (x - \beta)^4\left(A_1t^2 + B_1\right)\left(A_2t^2 + B_2\right)$$

and the integrand becomes

$$R_1(x)\left[\frac{(\alpha - \beta)^{-1}dt}{\sqrt{(A_1t^2 + B_1)\,(A_2t^2 + B_2)}}\right].$$

Finally, $R_1(x)$ can be written as $\pm(\alpha - \beta)R_3(t)$ where R_3 is a rational function of t.

Lemma 3.3.1. *There exist rational functions* R_4 *and* R_5 *such that* $R_3(t) + R_3(-t) = 2\,R_4(t^2)$ *and* $R_3(t) - R_3(-t) = 2t\,R_5(t^2)$. *Therefore,* $R_3(t) = R_4(t^2) + t\,R_5(t^2)$.

Exercise 3.3.2. *Use Mathematica or Maple to verify Lemma 3.3.1 for several rational functions of your choice, including at least one with arbitrary coefficients. Mathematica commands which might be useful are Denominator, Expand, Numerator, and Together. Then prove the lemma. [Hint: Check for even and odd functions.]*

The integral $\int \frac{R_1(x)}{y}\, dx$ is now reduced to

$$\int \frac{R_4(t^2)\, dt}{\sqrt{(A_1t^2 + B_1)\,(A_2t^2 + B_2)}} + \int \frac{t\,R_5(t^2)\, dt}{\sqrt{(A_1t^2 + B_1)\,(A_2t^2 + B_2)}}.$$

The substitution $u = t^2$ allows the second integral to be evaluated in terms of elementary functions. If $R_4(t^2)$ is expanded in partial fractions[1] the first integral is reduced to sums of integrals of the following type:

$$\int t^{2m}\left[\left(A_1t^2 + B_1\right)\left(A_2t^2 + B_2\right)\right]^{-1/2} dt \tag{3.3.3}$$

$$\int (1 + Nt^2)^{-m}\left[\left(A_1t^2 + B_1\right)\left(A_2t^2 + B_2\right)\right]^{-1/2} dt \tag{3.3.4}$$

[1] The Apart command in Mathematica does this.

where in (3.3.3) m is an integer, and in (3.3.4) m is a positive integer and $N \neq 0$. Reduction formulas can be derived to reduce (3.3.3) or (3.3.4) to a combination of known functions and integrals in the following canonical forms:

$$\int \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt, \tag{3.3.5}$$

$$\int t^2 \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt, \tag{3.3.6}$$

$$\int (1 + N t^2)^{-1} \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt. \tag{3.3.7}$$

Equations (3.3.5), (3.3.6), and (3.3.7) are the *elliptic integrals of the first, second, and third kinds*, so named by Legendre.

Exercise 3.3.3. *By differentiating* $t\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}$*, obtain a reduction formula for (3.3.3) when* $m = 2$*. Do you see how this process can be extended to all positive* m*?*

Exercise 3.3.4. *Find a reduction formula for (3.3.3) when* $m = -1$*.*

Exercise 3.3.5. *Show that the transformation* $t = \sin\theta$ *applied to (3.1.4), (3.1.6), or (3.1.8) yields integrals of the form (3.3.5), (3.3.6), and (3.3.7).*

For real elliptic integrals, all the essentially different combinations of signs in the radical are given in the following table.

A_1	+	+	−	+	+	−
B_1	+	−	+	−	−	+
A_2	+	+	+	+	−	−
B_2	+	+	+	−	+	+

Table 3.3.1.

Example 3.3.2. *Find the appropriate substitution for the second column of Table 3.3.1. Assume that* A_1*,* B_1*,* A_2*, and* B_2 *are all positive, so the radical is* $\sqrt{(A_1t^2 - B_1)(A_2t^2 + B_2)}$*. Substitute* $t = \sqrt{\frac{B_1}{A_1}}\sec\theta$*,* $dt = \sqrt{\frac{B_1}{A_1}}\sec\theta\tan\theta\, d\theta$ *so that*

$$\frac{dt}{\sqrt{(A_1t^2 - B_1)(A_2t^2 + B_2)}} = \frac{d\theta}{\sqrt{A_2B_1 + A_1B_2\cos^2\theta}} = \frac{1}{\sqrt{A_2B_1 + A_1B_2}} \frac{d\theta}{\sqrt{1 - \frac{B_2A_1}{B_2A_1 + B_1A_2}\sin^2\theta}}.$$

Here, $m = k^2 = \frac{B_2A_1}{B_2A_1 + B_1A_2} < 1$*, and we have the form of (3.1.4) or (3.1.5).*

Exercise 3.3.6. *Fill in the details of Example 3.3.2, and find the value of* $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{dt}{\sqrt{(2t^2 - 1)(t^2 + 2)}}$*.*

Exercise 3.3.7. *Pick another column of Table 3.3.1 and find the appropriate substitution. Consult with your classmates so that every column is considered by someone.*

3.4. The Jacobian Elliptic Functions

Trigonometric functions, even though they can do the job, are not the best things to use when reducing elliptic integrals. The ideal substitution in, say, (3.3.5), would be one where we have functions f, g, and h such that

$$t = f(v), \quad dt = g(v)\, h(v)\, dv, \quad A_1 t^2 + B_1 = g^2(v), \quad \text{and} \quad A_2 t^2 + B_2 = h^2(v).$$

Adjustments for the constants would have to be made, of course, but such a substitution would reduce (3.3.5) to $\int dv$. Fortunately, such functions exist! They are called the *Jacobian elliptic functions* and can be thought of as extensions of the trigonometric functions.

Consider the function defined by an integral as follows:

$$u = g(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^\phi d\theta = \phi = \sin^{-1} x.$$

(Here we used the substitution $t = \sin\theta$, $\;x = \sin\phi$, but we really don't want to focus on this part.) Assume that "sin" and "$\sin^{-1}$" are simply names we came up with for the functions involved here, making "$\sin^{-1}$" nothing more than another name for "g", which is *defined by the integral*. Thinking similarly, we can say that $g^{-1}(u) = \sin u$. Thus, we have *defined* the function "sin" as the inverse of the function g which is given in terms of the integral. A table of values for g can be calculated by numerical integration and this table with the entries reversed produces a table for g^{-1}. Now a new function can be defined by $h^{-1}(u) = \sqrt{1 - [g^{-1}(u)]^2}$. Clearly, $\left[g^{-1}(u)\right]^2 + \left[h^{-1}(u)\right]^2 = 1$, and we might want to give "h^{-1}" a new name, such as "cos", for instance. All of trigonometry can be developed in this way, without reference to angles, circles, or any of the usual stuff associated with trigonometry. We won't do trigonometry this way, because you already know that subject, but we will use this method to study the Jacobian elliptic functions.

In the spirit of the last paragraph, but assuming you know all about trigonometry, consider, for $0 \le m \le 1$,

$$u = \int_0^x \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\, dt = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \tag{3.4.1}$$

where the integrals are related by the substitution $t = \sin\theta$, $\;x = \sin\phi$. Note that, for fixed m, the integrals define u as a function of x or ϕ, and so a table of values for this function and its inverse can be constructed by numerical integration as discussed above.

Definition 3.4.1. *The Jacobian elliptic functions sn, cn, and dn are*

$$sn\, u = \sin\phi = x, \quad cn\, u = \cos\phi = \sqrt{1-x^2}, \quad dn\, u = \sqrt{1 - m\sin^2\phi} = \sqrt{1 - m\,x^2}$$

where u and ϕ or x are related by (3.4.1). In particular, sn is the inverse of the function of x defined by the first integral in (3.4.1). Sometimes specific dependence on the parameter m is denoted by $sn(u\,|\,m)$, $cn(u\,|\,m)$, and $dn(u\,|\,m)$.

Some immediate consequences of Definition 3.4.1 are, for any m,

$$sn^2 u + cn^2 u = 1, \quad m\, sn^2 u + dn^2 u = 1, \quad sn(0) = 0, \quad cn(0) = dn(0) = 1$$

Exercise 3.4.1. *Show that sn is an odd function and cn is an even function.*

The following notation and terminology applies to all Jacobian elliptic functions:

$$\begin{aligned} u &: \text{ argument} \\ m &: \text{ parameter} \\ m_1 = 1 - m &: \text{ complementary parameter} \\ \phi &: \text{ amplitude, denoted } \phi = am\, u \\ k = \sqrt{m} &: \text{ modulus} \\ k' = \sqrt{1-k^2} &: \text{ complementary modulus} \end{aligned}$$

Exercise 3.4.2. *Show that* $\frac{d\phi}{du} = dn\, u$, *and that* $\frac{d}{du} sn\, u = cn\, u\, dn\, u$. *Also find* $\frac{d}{du} cn\, u$ *and* $\frac{d}{du} dn\, u$.

Exercise 3.4.3. *Derive Formulas* **17.4.44**, **17.4.45**, *and* **17.4.52** *on page 596 of A&S. This verifies that sn, cn, and dn are among the kinds of functions we wanted at the beginning of this section.*

Exercise 3.4.4. *Show that* $sn(u\,|\,0) = \sin u$, $cn(u\,|\,0) = \cos u$, *and* $dn(u\,|\,0) = 1$. *Also show that* $sn(u\,|\,1) = \tanh u$ *and* $cn(u\,|\,1) = dn(u\,|\,1) = sech\, u$.

There are nine other Jacobian elliptic functions, all of which can be expressed in terms of *sn*, *cn*, and *dn* in much the same way all other trig functions can be expressed in terms of sin and cos. The notation uses only the letters s, c, d, and n according to the following rules. First, quotients of two of *sn*, *cn*, and *dn* are denoted by the first letter of the numerator function followed by the first letter of the denominator function. Second, reciprocals are denoted by writing the letters of the function whose reciprocal is taken in reverse order. Thus

$$\begin{aligned} ns(u) &= \frac{1}{sn(u)}, & nc(u) &= \frac{1}{cn(u)}, & nd(u) &= \frac{1}{dn(u)} \\ sc(u) &= \frac{sn(u)}{cn(u)}, & sd(u) &= \frac{sn(u)}{dn(u)}, & cd(u) &= \frac{cn(u)}{dn(u)} \\ cs(u) &= \frac{cn(u)}{sn(u)}, & ds(u) &= \frac{dn(u)}{sn(u)}, & dc(u) &= \frac{dn(u)}{cn(u)} \end{aligned} \tag{3.4.2}$$

Quotients of any two Jacobian elliptic functions can be reduced to a quotient involving *sn*, *cn*, and/or *dn*. For example

$$\frac{sc(u)}{sd(u)} = sc(u) \cdot ds(u) = \frac{sn(u)}{cn(u)} \cdot \frac{dn(u)}{sn(u)} = \frac{dn(u)}{cn(u)} = dc(u).$$

3.5. Addition Theorems

The Jacobian elliptic functions turn out to be not only periodic, but doubly periodic. Viewed as functions of a complex variable, they exhibit periodicity in both the real and imaginary directions. The following theorem will be quite useful in the next section for studying periodicity. Think about this theorem in the light of Exercise 3.4.4.

Theorem 3.5.1 (Addition Theorem). *For a fixed m,*

$$sn(u+v) = \frac{sn(u)\, cn(v)\, dn(v) + sn(v)\, cn(u)\, dn(u)}{1 - m\, sn^2(u)\, sn^2(v)}. \tag{3.5.1}$$

Proof. Let α be a constant, and suppose u and v are related by $u+v=\alpha$, so that $\dfrac{dv}{du}=-1$. Denote differentiation with respect to u by $\dot{}$, and let $s_1 = sn(u)$, $s_2 = sn(v)$. Then, keeping in mind that derivatives of s_2 require the chain rule,

$$\dot{s}_1^2 = (1-s_1^2)(1-m\,s_1^2) \quad \text{and} \quad \dot{s}_2^2 = (1-s_2^2)(1-m\,s_2^2).$$

If we differentiate again and divide by $2\dot{s}_1$ and $2\dot{s}_2$, we get

$$\ddot{s}_1 = -(1+m)s_1 + 2m\,s_1^3 \quad \text{and} \quad \ddot{s}_2 = -(1+m)s_2 + 2m\,s_2^3.$$

Thus

$$\frac{\ddot{s}_1 s_2 - \ddot{s}_2 s_1}{\dot{s}_1^2 s_2^2 - \dot{s}_2^2 s_1^2} = \frac{-2m\,s_1 s_2}{1-m\,s_1^2 s_2^2}$$

and multiplying both sides by $\dot{s}_1 s_2 + \dot{s}_2 s_1$ gives

$$\frac{\frac{d}{du}(\dot{s}_1 s_2 - \dot{s}_2 s_1)}{\dot{s}_1 s_2 - \dot{s}_2 s_1} = \frac{\frac{d}{du}(1-m\,s_1^2 s_2^2)}{1-m\,s_1^2 s_2^2}.$$

Integration and clearing of logarithms gives

$$\frac{\dot{s}_1 s_2 - \dot{s}_2 s_1}{1-m\,s_1^2 s_2^2} = \frac{sn(u)\,cn(v)\,dn(v) + sn(v)\,cn(u)\,dn(u)}{1-m\,sn^2(u)\,sn^2(v)} = C. \tag{3.5.2}$$

Equation (3.5.2) may be thought of as a solution of the differential equation $du + dv = 0$. But $u+v=\alpha$ is also a solution of this differential equation, so the two solutions must be dependent, i.e., there exists a function f such that

$$f(u+v) = \frac{sn(u)\,cn(v)\,dn(v) + sn(v)\,cn(u)\,dn(u)}{1-m\,sn^2(u)\,sn^2(v)}.$$

If $v=0$ we see from Exercise 3.4.4 that $f(u) = sn(u)$, so $f = sn$ and the theorem is proved. ♠

Exercise 3.5.1. *Derive the following addition theorems for cn and dn.*

$$cn(u+v) = \frac{cn(u)\,cn(v) - sn(u)\,sn(v)\,dn(u)\,dn(v)}{1-m\,sn^2(u)\,sn^2(v)} \tag{3.5.3}$$

$$dn(u+v) = \frac{dn(u)\,dn(v) - m\,sn(u)\,sn(v)\,cn(u)\,cn(v)}{1-m\,sn^2(u)\,sn^2(v)}. \tag{3.5.4}$$

3.6. Periodicity

Since $u = \displaystyle\int_0^x \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\,dt$, means that $sn(u\,|\,m) = x$, if we let $K = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\,dt$, then $sn(K\,|\,m) = 1$, $cn(K\,|\,m) = 0$, and $dn(K\,|\,m) = \sqrt{1-m} = k'$. By the addition theorem for sn, we get

$$sn(u+K) = \frac{sn(u)\,cn(K)\,dn(K) + sn(K)\,cn(u)\,dn(u)}{1-m\,sn^2(u)\,sn^2(K)} = \frac{cn(u)}{dn(u)} = cd(u).$$

Similarly, the addition theorems for cn and dn give

$$cn(u+K) = \frac{cn(u)\,cn(K) - sn(u)\,sn(K)\,dn(u)\,dn(K)}{1-m\,sn^2(u)\,sn^2(K)} = -\sqrt{1-m}\,sd(u)$$

$$dn(u+K) = \frac{dn(u)\,dn(K) - m\,sn(u)\,sn(K)\,cn(u)\,cn(K)}{1-m\,sn^2(u)\,sn^2(K)} = \sqrt{1-m}\,nd(u).$$

These results can be used to get

$$sn(u+2K) = -sn(u) \qquad cn(u+2K) = -cn(u) \qquad dn(u+2K) = dn(u)$$

and

$$sn(u+4K) = sn(u) \qquad cn(u+4K) = cn(u).$$

Thus, sn and cn have period $4K$, and dn has period $2K$, where $K = K(m)$ is the elliptic integral of the first kind. Note that we say "a period" instead of "the period" in the following theorem.

Theorem 3.6.1 (First Period Theorem). *If* $K = \int_0^1 \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\,dt = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$*, then* $4K$ *is a period of* sn *and* cn*, and* $2K$ *is a period of* dn*.*

Exercise 3.6.1. *Fill in the details of the proof of Theorem 3.6.1.*

To study other possible periods, let $K' = \int_0^1 \frac{1}{\sqrt{(1-t^2)\,(1-m_1t^2)}}\,dt$. K' is the same function of m_1 as K is of m. Suppose $0 < m < 1$ so that both k and k' are also strictly between 0 and 1.

Exercise 3.6.2. *In the integral for* K'*, substitute* $s^2 = \frac{1}{1-m_1t^2}$ *to get* $K' = \int_1^{\frac{1}{k}} \frac{1}{\sqrt{(s^2-1)\,(1-k^2t^2)}}\,ds.$

Now consider the integral $\int_0^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)\,(1-k^2s^2)}}\,dt$. Since $\frac{1}{k} > 1$, the integrand has a singularity at $t = 1$, and the integral has complex values for $1 < t < \frac{1}{k}$. To deal with these problems, we consider t a complex variable and look at the following path from the origin to $(\frac{1}{k}, 0)$ in the complex plane:

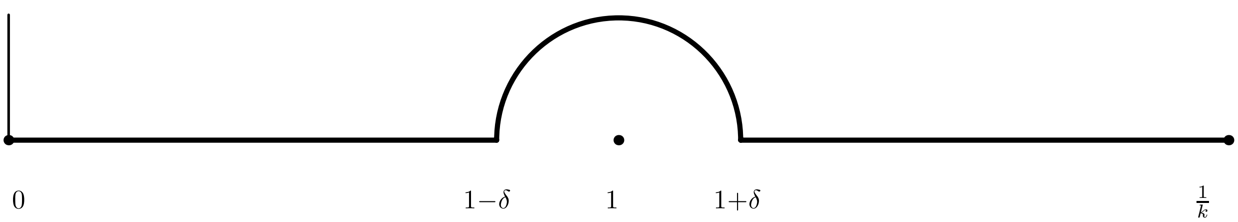


Figure 3.6.1

When t is on the semicircle near $1+\delta$, $t = 1+\delta e^{i\epsilon}$ for some $\epsilon > 0$. This makes $1-t^2 = \delta(2+\delta e^{i\epsilon})e^{i(\pi+\epsilon)}$. In order to get the principal value of the square root, we must replace $e^{i(\pi+\epsilon)}$ by $e^{-i(\pi-\epsilon)}$. Then, as $\epsilon \to 0+$, so that $t \to 1+\delta$ clockwise on the semicircle, we get

$$\begin{aligned}\sqrt{1-t^2} &= \lim_{\epsilon\to 0+} \sqrt{\delta(2+\delta e^{i\epsilon})}e^{i(\pi+\epsilon)/2} \\ &= \sqrt{\delta(2+\delta)}e^{-i\pi/2} \\ &= -i\sqrt{t^2-1}.\end{aligned}$$

We now can see that

$$\begin{aligned}\int_0^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)\,(1-k^2s^2)}}\,dt &= \int_0^1 \frac{1}{\sqrt{(1-t^2)\,(1-k^2s^2)}}\,dt + \int_1^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)\,(1-k^2s^2)}}\,dt \\ &= K(m) - \frac{1}{i}\int_1^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)\,(1-k^2s^2)}}\,dt \\ &= K(m) + i\,K'(m).\end{aligned}$$

Therefore,

$$sn(K+i\,K') = 1/k, \text{ from which we get } dn(K+i\,K') = 0$$

and

$$cn(K+i\,K') = \lim_{x\to 1/k}\sqrt{1-x^2} = \lim_{x\to 1/k}\left[-i\sqrt{x^2-1}\right] = -i\,\frac{k'}{k},$$

where the limits are taken along the path shown in Figure 3.6.1.

Exercise 3.6.3. *Show that*

$$sn(u+K+i\,K') = \frac{1}{k}\,dc(u), \qquad cn(u+K+i\,K') = -\frac{i\,k'}{k}\,nc(u), \qquad dn(u+K+i\,K') = i\,k'\,sc(u).$$

Exercise 3.6.4. *Show that*

$$sn(u+2K+2i\,K') = -sn(u), \qquad cn(u+2K+2i\,K') = cn(u), \qquad dn(u+2K+2i\,K') = -dn(u).$$

Exercise 3.6.5. *Show that*

$$sn(u+4K+4i\,K') = sn(u), \qquad dn(u+4K+4i\,K') = dn(u).$$

So, the Jacobian elliptic functions also have a complex period.

Theorem 3.6.2 (Second Period Theorem). *If* $K = \displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-mt^2)}}$, *and* $K' = \displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-m_1t^2)}}$, *then* $4K+4i\,K'$ *is a period of* sn *and* dn, *and* $2K+2i\,K'$ *is a period of* cn.

Theorem 3.6.3 (Third Period Theorem). *If* $K' = \displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-m_1t^2)}}$, *then* $4i\,K'$ *is a period of* cn *and* dn, *and* $2i\,K'$ *is a period of* sn.

Exercise 3.6.6. *Prove Theorem 3.6.3. [Hint:* $i\,K' = -K + K + i\,K'$.*]*

The numbers K and iK' are called the real and imaginary *quarter periods* of the Jacobian elliptic functions.

3.7. Zeros, Poles, and Period Parallelograms

Series expansions of sn, cn, and dn around $u=0$ are

$$\begin{aligned}
sn(u\,|\,m) &= u - \frac{1+m}{6}u^3 + \frac{1+14m+m^2}{120}u^5 + O(u^7)\\
cn(u\,|\,m) &= 1 - \frac{1}{2}u^2 + \frac{1+4m}{24}u^4 + O(u^6)\\
dn(u\,|\,m) &= 1 - \frac{m}{2}u^2 + \frac{4m+m^2}{24}u^4 + O(u^6)
\end{aligned}$$

Series expansions of sn, cn, and dn around $u = i\,K'$ are

$$\begin{aligned}
sn(u+i\,K'\,|\,m) &= \frac{1}{k\,sn(u\,|\,m)} = \frac{1}{k}\left[\frac{1}{u} + \frac{1+m}{6}u + \frac{7-22m+7m^2}{360}u^3 + O(u^4)\right]\\
cn(u+i\,K'\,|\,m) &= \frac{-i}{k}ds(u\,|\,m) = \frac{-i}{k}\left[\frac{1}{u} + \frac{1-2m}{6}u + \frac{7+8m-8m^2}{360}u^3 + O(u^4)\right]\\
dn(u+i\,K'\,|\,m) &= -i\,cs(u\,|\,m) = -i\left[\frac{1}{u} + \frac{-2+m}{6}u + \frac{-8+8m+7m^2}{360}u^3 + O(u^4)\right]
\end{aligned}$$

From these expansions we can identify the poles and residues of the Jacobian elliptic functions.

Theorem 3.7.1 (First Pole Theorem). *At the point $u = i\,K'$ the functions sn, cn, and dn have simple poles with residues $1/k$, $-i/k$, and $-i$ respectively.*

Theorem 3.7.2 (Second Pole Theorem). *At the point $u = 2K + i\,K'$ the functions sn and cn have simple poles with residues $-1/k$ and i/k. At the point $u = 3i\,K'$ the function dn has a simple pole with residue i.*

Exercise 3.7.1. *Prove both Pole Theorems. If you want to verify the series expansions, use Mathematica or Maple.*

From the period theorems we see that each Jacobian elliptic function has a smallest *period parallelogram* in the complex plane. It is customary to translate period parallelograms so that no zeros or poles are on the boundary. When this is done, we see that the period parallelogram for each Jacobian elliptic function contains exactly two zeros and two poles in its interior. Unless stated otherwise, we shall assume that any period parallelogram has been translated in this manner. See Figure 3.7.1, in which zeros are indicated by a o and poles by a $*$.

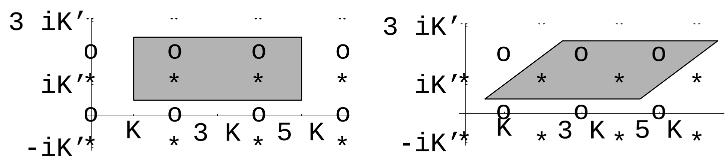


Figure 3.7.1. Period parallelograms for *sn* and *cn*.

Exercise 3.7.2. *Sketch period parallelograms for dn, sc, and cd with no zeros or poles on the boundaries.*

If C denotes the counterclockwise boundary of a period parallelogram for some Jacobian elliptic function, the Residue Theorem from complex analysis says that the the integral around C of the Jacobian elliptic function is $1/2\pi i$ times the sum of the residues at the two poles inside C. In the next section we will see that for elliptic functions in general this integral is zero.

Exercise 3.7.3. *If C denotes the boundary of a period parallelogram, oriented counterclockwise, compute $\int_C sn(u)\,du$, $\int_C cn(u)\,du$, and $\int_C dn(u)\,du$.*

A useful device for dealing with the Jacobian elliptic functions is the doubly infinite array, or lattice, consisting of the letters s, c, d, and n shown in Figure 3.7.2. Think of this lattice in the complex plane and denote one of the points labelled s by K_s. Then denote the point to the east labelled c by K_c, the point to the north labelled n by K_n, and the point to the southwest labelled d by K_d. If we put the origin at K_s, then the sum (of complex numbers, or of vectors) $K_s + K_c + K_d + K_n = 0$. Assume the scale on the lattice is such that $K_c = K$, $K_n = iK'$, and $K_d = -K - iK'$, where K and iK' are the real and imaginary quarter periods.

s	c	s	c	s	c	s
n	d	n	d	n	d	n
s	c	s	c	s	c	s
n	d	n	d	n	d	n

Figure 3.7.2. This pattern is repeated indefinitely on all sides.

If the letters p, q, r, and t are any permutation of s, c, d, and n, then the Jacobian elliptic function *pq* has the following properties. See also A&S, p.569.

(1) *pq* is doubly periodic with a simple zero at K_p and a simple pole at K_q.

(2) The step $K_q - K_p$ from the zero to the pole is a half-period; the numbers K_c, K_n, and K_d not equal to $K_q - K_p$ are quarter-periods.

(3) In the series expansion of *pq* around $u = 0$ the coefficient of the leading term is 1.

Here are plots of the *modular surfaces* over one period parallelogram of the functions $w = sn(u\,|\,\frac{1}{2})$ and $w = dn(u\,|\,\frac{1}{2})$, where $u = x + i\,y$. Complex functions of a complex variable require four dimensions for a complete graph, but a useful compromise is to plot the modulus, or absolute value of a complex function, which is essentially a real function of the real and imaginary parts of the complex variable. The surfaces in Figure 3.7.3 are plots of $w = |\,sn(x + i\,y\,|\,\frac{1}{2})\,|$ and $w = |\,dn(x + i\,y\,|\,\frac{1}{2})\,|$. The depressions correspond to zeros and the towers correspond to poles.

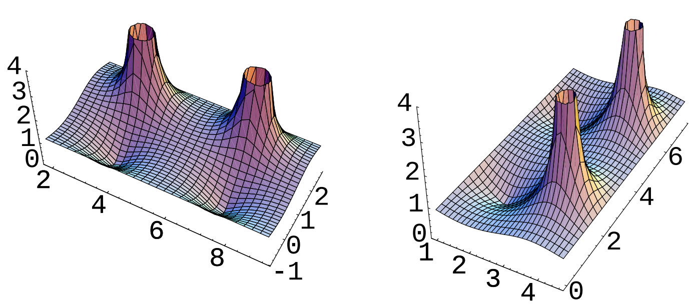


Figure 3.7.3. Modular surfaces of *sn* and *dn*.

Exercise 3.7.4. *Use the lattice in Figure 3.7.2 to determine the periods, zeros, and poles of cd and ds. Use Mathematica or Maple to plot the modular surfaces over one period parallelogram.*

Exercise 3.7.5. *Why are the functions sn, cn, and dn called "the Copolar Trio" in A&S, 16.3?*

3.8. General Elliptic Functions

Definition 3.8.1. *An elliptic function of a complex variable is a doubly periodic function which is meromorphic, i.e., analytic except for poles, in the finite complex plane.*

If the smallest periods of an elliptic function are $2\omega_1$ and $2\omega_2$, then the parallelogram with vertices 0, $2\omega_1$, $2\omega_1 + 2\omega_2$, and $2\omega_2$ is the *fundamental period parallelogram.* A *period parallelogram* is any translation of a fundamental period parallelogram by integer multiples of $2\omega_1$ and/or $2\omega_2$. A *cell* is a translation of a period parallelogram so that no poles are on the boundary. Parts of the proofs of the next few theorems depend on the theory of functions of a complex variable. If you have not studied that subject (or if you have forgotten it), learn what the theorems say, and come back to the proofs after you have studied the theory of functions of a complex variable. In fact, this material should be an incentive to take a complex variables course!

Theorem 3.8.1. *An elliptic function has a finite number of poles in any cell.*

Outline of Proof. A cell is a bounded set in the complex plane. If the number of poles in a cell is infinite, then by the two-dimensional Bolzano-Weierstrass Theorem, the poles would have a limit point in the cell. This limit point would be an essential singularity of the elliptic function. It could not be a pole, because a pole is an isolated singularity. This is a contradiction of the definition of an elliptic function. ♠

Theorem 3.8.2. *An elliptic function has a finite number of zeros in any cell.*

Proof. If not, then the reciprocal function would have an infinite number of poles in a cell, and as in the proof of Theorem 3.8.1, would have an essential singularity in the cell. This point would also be an essential singularity of the original function, contradicting the definition of an elliptic function. ♠

Theorem 3.8.3. *In a cell, the sum of the residues at the poles of an elliptic function is zero.*

Proof. Let C be the boundary of the cell, oriented counterclockwise, and let the vertices be given by t, $t+2\omega_1$, $t+2\omega_1+2\omega_2$, and $t+2\omega_2$, where $2\omega_1$ and $2\omega_2$ are the periods. Call the elliptic function f. The sum of the residues is

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\left[\int_t^{t+2\omega_1} f(z)\,dz + \int_{t+2\omega_1}^{t+2\omega_1+2\omega_2} f(z)\,dz + \int_{t+2\omega_1+2\omega_2}^{t+2\omega_2} f(z)\,dz + \int_{t+2\omega_2}^{t} f(z)\,dz\right].$$

In the second integral, replace z by $z + 2\omega_1$, and in the third integral, replace z by $z + 2\omega_2$. We then get

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_t^{t+2\omega_1} (f(z) - f(z+2\omega_2))\,dz - \frac{1}{2\pi i}\int_t^{t+2\omega_2} (f(z) - f(z+2\omega_1))\,dz$$

By the periodicity of f, each of these integrals is zero. ♠

Theorem 3.8.4 (Liouville's Theorem for Elliptic Functions). *An elliptic function having no poles in a cell is a constant.*

Proof. If f is elliptic having no poles in a cell, then f is analytic both in the cell and on the boundary of the cell. Thus, f is bounded on the closed cell, and so there is an M such that for z in the closed cell, $|f(z)| < M$. By periodicity, we then have that $|f(z)| < M$ for all z in the complex plane, and so by Liouville's (other, more famous) Theorem, f is constant. ♠

Definition 3.8.2. *The order of an elliptic function f is equal to the number of poles, counted according to multiplicity, in a cell.*

The following lemma is useful in determining the order of elliptic functions.

Lemma 3.8.1. *If f is an elliptic function and z_0 is any complex number, the number of roots of the equation $f(z) = z_0$ in any cell depends only on f and not on z_0.*

Proof. Let C be the boundary, oriented counterlockwise, of a cell, and let Z and P be the respective numbers of zeros and poles, counted according to multiplicity, of $f(z)-z_0$ in the cell. Then by the Argument Principle,

$$Z - P = \frac{1}{2\pi\, i}\int_C \frac{f'(z)}{f(z) - z_0}\, dz.$$

Breaking the integral into four parts and substituting as in the proof of Theorem 3.8.3, we get $Z - P = 0$. Thus, $f(z) - z_0$ has the same number of zeros as poles; but the number of poles is the same as the number of poles of f, which is independent of z_0. ♠

Definition 3.8.3. *The order of an elliptic function f is equal to the number of zeros, counted according to multiplicity, of f in any cell.*

Exercise 3.8.1. *Prove that if f is a nonconstant elliptic function, then the order of f is at least two.*

Thus, in terms of the number of poles, the simplest elliptic functions are those of order two, of which there are two kinds: (1) those having a single pole of order two whose residue is zero, and (2) those having two simple poles whose residues are negatives of one another. The Jacobian elliptic functions are of the second kind. An example of an elliptic function of the first kind will be given in the next section.

3.9. Weierstrass' P-function

Let ω_1 and ω_2 be two complex numbers such that the quotient ω_1/ω_2 is not a real number, and for integers m and n let $\Omega_{m,n} = 2m\,\omega_1 + 2n\,\omega_2$. Then Weierstrass' P-function is defined by

$$P(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left[\frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\right], \tag{3.9.1}$$

where the sum is over all integer values of m and n, and the prime notation indicates that m and n simultaneously zero is not included. The series for $P(z)$ can be shown to converge absolutely and uniformly except at the points $\Omega_{m,n}$, which are poles.

Exercise 3.9.1. *Show that $P'(z) = -2\sum_{m,n}\dfrac{1}{(z - \Omega_{m,n})^3}$. Note the absence of the prime on the sum.*

Exercise 3.9.2. *Show that P is an even function and that P' is an odd function.*

Theorem 3.9.1. *$P'(z)$ is doubly periodic with periods $2\omega_1$ and $2\omega_2$, and therefore P' is an elliptic function.*

Proof. The sets $\{\Omega_{m,n}\}$, $\{\Omega_{m,n} - 2\omega_1\}$, and $\{\Omega_{m,n} - 2\omega_2\}$ are all the same. ♠

Theorem 3.9.2. *P is an elliptic function with periods $2\omega_1$ and $2\omega_2$.*

Proof. Since $P'(z + 2\omega_1) = P'(z)$, we have $P(z + 2\omega_1) = P(z) + A$, where A is a constant. If $z = -\omega_1$, we have $P(\omega_1) = P(-\omega_1) + A$, and since P is an even function, $A = 0$. Similarly, $P(z + 2\omega_2) = P(z)$. ♠

$\tilde{P}(z) = P(z) - z^{-2}$ is analytic in a neighborhood of the origin and is an even function, so $\tilde{P}$ has a series expansion around $z = 0$:

$$\begin{aligned}
\tilde{P}(z) &= a_2 z^2 + a_4 z^4 + O(z^6) \\
a_2 &= \frac{6}{2!} \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^4} = \frac{1}{20} g_2 \\
g_2 &= 60 \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^4} \\
a_4 &= \frac{120}{4!} \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^6} = \frac{1}{28} g_3 \\
g_3 &= 140 \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^6}
\end{aligned}$$

Thus we get the following series representations.

$$\begin{aligned}
P(z) &= z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6) \\
P'(z) &= -2z^{-3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + O(z^5) \\
P(z)^3 &= z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + O(z^2) \\
(P'(z))^2 &= 4z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + O(z^2)
\end{aligned}$$

Combining these leads to

$$(P'(z))^2 - 4\,P^3(z) + g_2 P(z) + g_3 = O(z^2). \tag{3.9.1}$$

Exercise 3.9.3. *Verify the details in the derivation of equation (3.9.1).*

The left side of (3.9.1) is an elliptic function with periods the same as P and is analytic at $z = 0$. By periodicity, then, it is analytic at each of the points $\Omega_{m,n}$. But the points $\Omega_{m,n}$ are the only points where the left side of (3.9.1) can have poles, so it is an elliptic function with no poles, and hence a constant. Let $z \to 0$ to see that the constant is 0.

Exercise 3.9.4. *Verify the statements in the last paragraph.*

The numbers g_2 and g_3 are called the *invariants* of P. P satisfies the differential equation

$$(P'(z))^2 = 4\,P^3(z) - g_2 P(z) - g_3. \tag{3.9.2}$$

3.10. Elliptic Functions in Terms of P and P'

Suppose f is an elliptic function and let P be the Weierstrass elliptic function (WEF) with the same periods as f. Then

$$\begin{aligned}
f(z) &= \frac{1}{2}\left[f(z) + f(-z)\right] + \frac{1}{2}\left[\left(f(z) - f(-z)\right)\left(P'(z)\right)^{-1}\right] P'(z) \\
&= (\text{even elliptic function}) + (\text{even elliptic function}) P'(z)
\end{aligned}$$

So, if we can express any *even* elliptic function in terms of P and P', then we can so express any elliptic function.

Suppose ϕ is an even elliptic function. The zeros and poles of ϕ in a cell can each be arranged in two sets:

zeros: $\{a_1, a_2, \ldots, a_n\}$ and additional points in the cell congruent to $\{-a_1, -a_2, \ldots, -a_n\}$.

poles: $\{b_1, b_2, \ldots, b_n\}$ and additional points in the cell congruent to $\{-b_1, -b_2, \ldots, -b_n\}$.

Consider the function

$$G(z) = \frac{1}{\phi(z)} \prod_{j=1}^{n} \frac{(P(z) - P(a_j))^?}{(P(z) - P(b_j))^?}.$$

Exercise 3.10.1. *Prove that G is a constant function when the question marks are suitably replaced. (For a specific case, see Example 3.10.1 below.)*

Thus, $\phi(z) = A \prod_{j=1}^{n} \frac{(P(z) - P(a_j))^?}{(P(z) - P(b_j))^?}$, and we have the following theorem.

Theorem 3.10.1. *Any elliptic function can be expressed in terms of the WEFs P and P' with the same periods. The expression will be rational in P and linear in P'.*

A related theorem is the following.

Theorem 3.10.2. *An algebraic (polynomial, I believe - LMH) relation exists between any two elliptic functions with the same periods.*

Outline of proof: Let f and ϕ be elliptic with the same periods. By Theorem 3.10.1, each can be expressed as a rational function of the WEFs P and P' having the same periods, say $f(z) = R_1(P(z), P'(z))$, $\phi(z) = R_2(P(z), P'(z))$. We can get an algebraic relation between f and ϕ by eliminating P and P' from these equations plus (3.9.2). ♠

Corollary 3.10.1. *Every elliptic function is related to its derivative by an algebraic relation.*

Proof: Clear, and left to the reader.

Example 3.10.1. *Express $cn\, z$ in terms of P and P'. Since cn is even, has periods $4K$ and $2K + 2iK'$, has zeros at K and $3K$, and has poles at iK' and $2K + iK'$, we can let $a_1 = K$ and $b_1 = iK'$. Thus,*

$$A = \frac{1}{cn\, z} \frac{P(z) - P(K)}{P(z) - P(iK')}.$$

As $z \to 0$ we see that $A = 1$, so

$$cn\, z = \frac{P(z) - P(K)}{P(z) - P(iK')}.$$

Exercise 3.10.2. *Let $m = .5$ and verify all statements in Example 3.10.1. Compare modular surface plots of cn and its representation in terms of P.*

The algebraic relation between two equiperiodic elliptic functions depends on the orders of the elliptic functions. Recalling Lemma 3.8.1, if f has order m and ϕ has order n, then corresponding to any value of $f(z)$, there are m values of z. Corresponding to each of these m values of z there are m values of $\phi(z)$. Similarly, to each value of $\phi(z)$ there correspond n values of $f(z)$. Thus, the algebraic relation between f and ϕ will be of degree m or lower in ϕ and degree n or lower in f.

Example 3.10.2. *The functions $f(z) = P(z)$ and $\phi(z) = P^2(z)$ have orders 2 and 4, so their relation will be of degree at most 2 in ϕ and at most 4 in f. The relation between them is obviously $\phi = f^2$, of less than maximum degrees.*

Example 3.10.3. *Let $f(z) = P(z)$ (order 2) and $\phi(z) = P'(z)$ (order 3). Their relation is given by equation (3.9.2), and is of degree 2 in ϕ and degree 3 in f.*

3.11. Elliptic Wheels - An Application

The material in this section is taken from: Leon Hall and Stan Wagon, Roads and wheels, *Mathematics Magazine* 65, (1992), 283-301. See this article for more details.

Suppose we are given a wheel in the form of a function defined by $r = g(\theta)$ in polar coordinates with the axle of the wheel at the origin, or pole. The problem is, what road is required for this wheel to roll on so that the axle remains level? The axle may or may not coincide with the wheel's geometric center, and the road is assumed to provide enough friction so the wheel never slips. Assume the road (to be found) has equation $y = f(x)$.

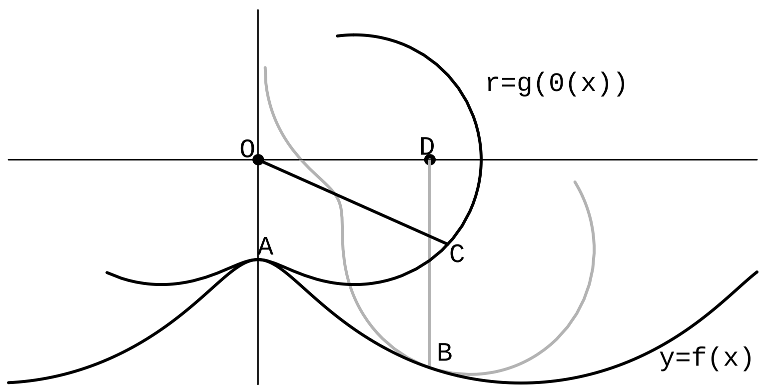


Figure 3.11.1. Wheel - road relationships

Three conditions will guarantee that the axle of the wheel moves horizontally on the x-axis as the wheel rolls on the road. These conditions are illustrated in Figure 3.11.1. First, the initial point of contact must be directly below the origin, which means that when $x = 0$, $\theta = -\pi/2$. Second, corresponding arc lengths along the road and on the wheel must be equal. In Figure 3.11.1, this means that the road length from A to B must equal the wheel length from A to C. Third, the radius of the wheel must match the depth of the road at the corresponding point, which means that $OC = DB$, or $g(\theta(x)) = -f(x)$. The arc length condition gives

$$\int_0^x \sqrt{1 + f'(u)^2}\, du = \int_{-\pi/2}^{\theta} \sqrt{g(\phi)^2 + g'(\phi)^2}\, d\phi.$$

Differentiation with respect to x and simplification leads to the initial value problem

$$\frac{d\theta}{dx} = \frac{1}{g(\theta)}, \qquad \theta(0) = -\pi/2,$$

whose solution expresses θ as a function of x. The road is then given by $y = -g(\theta(x))$.

Exercise 3.11.1. *Fill in the details of the derivation sketched above.*

Example 3.11.1. *Consider the ellipse with polar equation $r = \dfrac{k\,e}{1 - e\sin\theta}$, where e is the eccentricity of the ellipse and k is the distance from the origin to the corresponding directrix. The axle for this wheel is* *the focus which is initially at the origin. The details get a bit messy (guess who gets to do them!) but no special functions are required. The solution of the IVP turns out to be*

$$\frac{a\,x}{2\,k\,e} = \arctan\left(\frac{tan(\theta/2) - e}{a}\right) + \arctan\left(\frac{1+e}{a}\right),$$

where $a = \sqrt{1-e^2}$*. Now take the tangent of both sides and do some trig to get*

$$\frac{(1+e)(1-\cos^2(cx))}{(1-e)(1+\cos(cx))^2} = \frac{1+\sin\theta}{1-\sin\theta},$$

where $c = a/(ke)$*. Finally, solve for* $\sin\theta$ *and substitute into* $y(x) = \dfrac{k\,e}{1-e\,\sin\theta(x)}$ *to get the road*

$$y = -\frac{k\,e}{a^2}(1 - e\,\cos{(cx)}.$$

Thus, the road for an elliptic wheel with axle at a focus is essentially a cosine curve. Figure 3.11.2 illustrates the case $k=1$ *and* $e = 1/\sqrt{2}$*.*

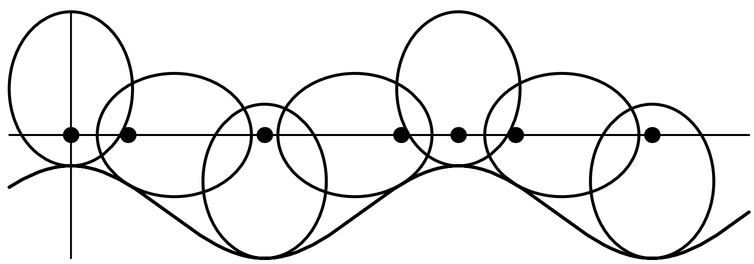


Figure 3.11.2. The axle at a focus yields a cosine road.

Exercise 3.11.2. *Fill in all the details in Example 3.11.1 for the case* $k=1$ *and* $e = 1/\sqrt{2}$*.*

Example 3.11.2. *We now consider a rolling ellipse where the axle is at the center of the ellipse. The ellipse* $x^2/a^2 + y^2/b^2 = 1$ *has polar representation* $r = b/\sqrt{1-m\,\cos^2\theta}$*, where* $m = 1 - b^2/a^2$ *(assume* $a > b$*). The IVP in* θ *and* x *is again separable and we get*

$$\int_{-\pi/2}^{\theta(x)} \frac{d\phi}{\sqrt{1-m\,\cos^2\phi}} = \int_0^x \frac{dt}{b}.$$

The substitution $\psi = \phi + \pi/2$ *yields*

$$\int_0^{\theta(x)+\pi/2} \frac{d\psi}{\sqrt{1-m\,\sin^2\psi}} = \frac{x}{b},$$

which involves an incomplete elliptic integral of the first kind. In terms of the Jacobian elliptic functions, we get $\sin{(\theta+\pi/2)} = sn(\frac{x}{b}\,|\,m)$*. The road is then*

$$y = \frac{-b}{dn(\frac{x}{b}\,|\,m)} = -b\;nd(\frac{x}{b}\,|\,\frac{a^2-b^2}{a^2}).$$

See Figure 3.11.3 for the $a=1$ *and* $b=1/2$ *case.*

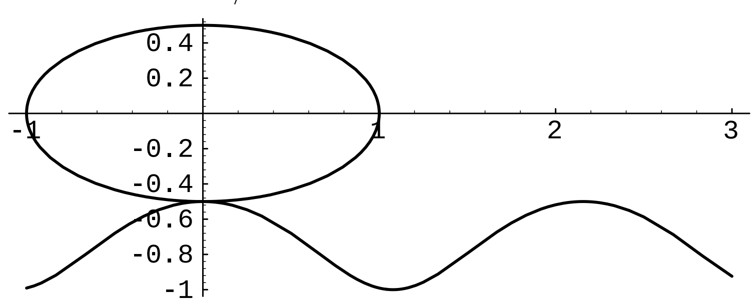


Figure 3.11.3. The axle at the center leads to Jacobian elliptic functions.

Exercise 3.11.3. *Fill in all the details in Example 3.11.2 for the case $a = 1$ and $b = 1/2$.*

Exercise 3.11.4. *Determine the road in terms of a Jacobian elliptic function for a center-axle elliptic wheel, $x^2/a^2 + y^2/b^2 = 1$, when $b > a$.*

3.12. Miscellaneous Integrals

Exercise 3.12.1. *Evaluate* $\displaystyle\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}$*, where $a > b$. (See A&S, p. 596.)*

Exercise 3.12.2. *Evaluate* $\displaystyle\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}$*, where $a > b$. (See A&S, p. 596.)*

Exercises 3.12.3-7. *Do Examples 8-12, A&S, pp.603-04.*

Chapter 4. Hypergeometric Functions

4.1. Solutions of Linear DEs at Regular Singular Points

Consider the differential equation

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0. \tag{4.1.1}$$

If either p or q has a singularity at $x = x_0$, then x_0 is a *singular point* of (4.1.1). The singular point x_0 is *regular* if both the limits

$$\lim_{x\to x_0} (x - x_0)p(x) \quad \text{and} \quad \lim_{x\to x_0} (x - x_0)^2 q(x)$$

exist. Call these limits, when they exist, p_0 and q_0. The *exponents at the regular singular point* x_0 of (4.1.1) are the roots of the *indicial equation*

$$r(r-1) + p_0 r + q_0 = 0.$$

If x_0 is a regular singular point of (4.1.1), then one solution is representable as a *Frobenius series* and has the form

$$y_1(x) = \sum_{k=0}^{\infty} a_k(r)(x - x_0)^{k+r} \tag{4.1.2}$$

where r is an exponent at x_0, i.e., a root of the indicial equation. The coefficients a_k can be found up to a constant multiple by substituting the series into (4.1.1) and equating coefficients. Unfortunately, we are only guaranteed one Frobenius series solution of (4.1.1), which is a second order linear homogeneous DE, and so has two linearly independent solutions. The second solution in the neighborhood of a regular singular point will take one of three forms.

Case 1. *If the exponents do not differ by an integer, then the second solution of (4.1.1) is found by using the other exponent in the series (4.1.2).*

Case 2. *If the exponents are equal, the second solution has the form*

$$y_2(x) = y_1(x)\, \log(x - x_0) + \sum_{k=1}^{\infty} b_k(r)\, (x - x_0)^{k+r},$$

where r is the exponent and y_1 is the solution given by (4.1.2).

Case 3. *If the exponents r_1 and r_2 differ by a positive integer, $r_1 - r_2 = N$, then one solution is given by (4.1.2) using $r = r_1$, and the second solution has the form*

$$y_2(x) = C\, y_1(x)\, \log(x - x_0) + \sum_{k=0}^{\infty} c_k(r_2)\, (x - x_0)^{k+r_2}.$$

The constant C may or may not be zero.

Example 4.1.1. *Legendre's differential equation is*

$$(1 - x^2)y''(x) - 2x\, y'(x) + n(n+1)\, y(x) = 0.$$

The most interesting case is when n is a nonnegative integer. At the regular singular point $x = 1$, the indicial equation is $r^2 = 0$, making the exponents at $x = 1$ equal to $0, 0$. For simplicity using Frobenius series, translate $x = 1$ to the origin by $x = u + 1$. The equivalent DE is

$$u(u+2)\, y''(u) + 2(u+1)\, y'(u) - n(n+1)\, y(u) = 0.$$

The regular singular point $u = 0$ corresponds to $x = 1$ and has the same exponents, both 0. The Frobenius series is $\sum_{k=0}^{\infty} a_k u^k$, *and substitution of the series into the DE yields*

$$\sum_{k=0}^{\infty} \left[(k+n+1)(k-n)a_k + 2(k+1)^2 a_{k+1}\right] u^k = 0.$$

Equating coefficients leads to the recurrence relation

$$a_{k+1} = \frac{-(k+n+1)(k-n)}{2(k+1)^2} a_k,$$

which gives

$$a_k = \frac{(-1)^k (n+1)_k (-n)_k}{2^k (k!)^2} a_0.$$

The the $(\cdot)_k$ notation represents the factorial function and is defined by $(z)_k = z(z+1)\cdots(z+k-1) = \Gamma(z+k)/\Gamma(z)$. The Frobenius series solution to Legendre's DE is, for $a_0 = 1$,

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{(1)_k k!} \left(\frac{1-x}{2}\right)^k.$$

Note that the series terminates if n is a nonnegative integer; the resulting polynomial is denoted $P_n(x)$, and is the Legendre polynomial of degree n. Also note that $P_n(1) = 1$ for all nonnegative integers n.

Exercise 4.1.1. *Fill in all the details and verify all the claims in Example 4.1.1. Get comfortable dealing with the factorial function.*

Example 4.1.2. *Bessel's differential equation is*

$$x^2\, y''(x) + x\, y'(x) + (x^2 - \nu^2)\, y(x) = 0.$$

The Frobenius series solution turns out to be

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k} k! (1+\nu)_k} a_0.$$

If we let $a_0 = \dfrac{1}{2^{\nu}\Gamma(\nu+1)}$, we get the "standard" solution to Bessel's DE, the Bessel funtion of the first kind of order ν, denoted by $J_\nu(x)$:

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{k!(1+\nu)_k} \left(\frac{-x^2}{4}\right)^k.$$

Exercise 4.1.2. *For Bessel's DE, show that $x = 0$ is a regular singular point with exponents $\pm\nu$, and fill in the details in the derivation of the formula for $J_\nu(x)$.*

4.2. Equations of Fuchsian Type

Consider the differential equation

$$y''(z) + p(z)\, y'(z) + q(z)\, y(z) = 0. \tag{4.2.1}$$

We call (4.2.1) an *equation of Fuchsian type* if every singular point is a regular singular point.

Lemma 4.2.1. *If (4.2.1) is of Fuchsian type, then the number of singular points of (4.2.1) is finite.*

Proof. At each singular point, either p or q has a pole. Suppose there are infinitely many singular points. Then either p or q has infinitely many poles. These poles have a limit point (possibly ∞) which is an essential singularity of p or q. But such an essential singularity corresponds to an irregular singular point of (4.2.1), contradicting the assumption that (4.2.1) is of Fuchsian type. ♠

Suppose (4.2.1) is of Fuchsian type and has exactly $m+1$ distinct singular points, where $m \geq 2$. Denote the singularities by $z = z_k, \quad k = 1, \ldots, m \quad$ and $\quad z = \infty$. Then p can have no singularities in the finite plane except poles of order one at the $z_k s$. So,

$$p(z) = \frac{p_1(z)}{(z - z_1)(z - z_2)\cdots(z - z_m)},$$

where p_1 is a polynomial. Also, q can have no singularities except poles of order $\leq$ two at the $z_k s$:

$$q(z) = \frac{q_1(z)}{(z - z_1)^2(z - z_2)^2\cdots(z - z_m)^2}.$$

The maximum degree of the polynomials p_1 and q_1 can be determined using the regular singular point at ∞. Let $z = 1/t$, giving

$$\frac{d^2y}{dt^2} + \frac{1}{t}\left[2 - \frac{p_1(\frac{1}{t})}{t(\frac{1}{t} - z_1)\cdots(\frac{1}{t} - z_m)}\right]\frac{dy}{dt} + \frac{1}{t^2}\left[\frac{q_1(\frac{1}{t})}{t^2(\frac{1}{t} - z_1)^2\cdots(\frac{1}{t} - z_m)^2}\right]y = 0. \tag{4.2.2}$$

In order for $z = \infty$, or $t = 0$, to be a regular singular point, the functions in the brackets in (4.2.2) must be analytic at $t = 0$. This means $degree(p_1) \leq m - 1$ and $degree(q_1) \leq 2m - 2$. Thus we have the following theorem.

Theorem 4.2.1. *If equation (4.2.1) is of Fuchsian type and has exactly $m+1$ distinct singular points, $z = z_k, \quad k = 1, \ldots, m \quad$ and $\quad z = \infty$, then (4.2.1) can be written*

$$y''(z) + \frac{T_{(m-1)}(z)}{\psi(z)}\,y'(z) + \frac{T_{(2m-2)}(z)}{\psi^2(z)}\,y(z) = 0,$$

where $\psi(z) = \prod_{k=1}^{m}(z - z_k)$ and $T_{(j)}(z)$ is a polynomial of degree at most j in z.

Corollary 4.2.1. *There exist constants A_k, $k = 1, 2, \ldots, m$ such that $p(z) = \sum_{k=1}^{m} \frac{A_k}{z - z_k}$.*

Corollary 4.2.2. *There exist constants B_k and C_k, $k = 1, 2, \ldots, m$, such that*

$$q(z) = \sum_{k=1}^{m}\left(\frac{B_k}{(z - z_k)^2} + \frac{C_k}{z - z_k}\right), \text{ and } \sum_{k=1}^{m} C_k = 0.$$

Exercise 4.2.1. *Prove both the above corollaries.*

Equation (4.2.1) thus can be written in the form

$$y''(z) + \sum_{k=1}^{m}\frac{A_k}{z - z_k}\,y'(z) + \sum_{k=1}^{m}\left(\frac{B_k}{(z - z_k)^2} + \frac{C_k}{z - z_k}\right)y(z) = 0, \tag{4.2.3}$$

where $\sum_{k=1}^{m} C_k = 0$.

Denote the exponents at the singular point z_k by $\alpha_{1,k}$ and $\alpha_{2,k}$, and the exponents at ∞ by $\alpha_{1,\infty}$ and $\alpha_{2,\infty}$. Since the indicial equation at z_k is $r^2 + (A_k - 1)r + B_k = 0$, we get

$$\alpha_{1,k} + \alpha_{2,k} = 1 - A_k \quad \text{and} \quad \alpha_{1,k}\alpha_{2,k} = B_k.$$

For the singularity at ∞, we get

$$\alpha_{1,\infty} + \alpha_{2,\infty} = -1 + \sum_{k=1}^{m} A_k \quad \text{and} \quad \alpha_{1,\infty}\alpha_{2,\infty} = \sum_{k=1}^{m} (B_k + C_k z_k).$$

Thus, the sum of all the exponents for all the singular points is

$$\alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{m} (\alpha_{1,k} + \alpha_{2,k}) = m - 1.$$

This number depends only on the number of singularities (and the order of the equation), and is the *Fuchsian invariant* for the second order DE of Fuchsian type. For the Fuchsian DE of order n, the Fuchsian invariant is $(m-1)n(n-1)/2$.

Example 4.2.1. *A second order Fuchsian DE with $m = 2$ contains five arbitrary parameters, A_1, A_2, B_1, B_2, and $C_1 = -C_2$. Also, there are six exponents, with sum one (Fuchsian invariant), such that*

$$\begin{aligned}
&A_1 = 1 - \alpha_{1,1} - \alpha_{2,1} \\
&A_2 = 1 - \alpha_{1,2} - \alpha_{2,2} \\
&B_1 = \alpha_{1,1}\alpha_{2,1} \\
&B_2 = \alpha_{1,2}\alpha_{2,2} \\
&B_1 + B_2 + C_1 z_1 + C_2 z_2 = \alpha_{1,\infty}\alpha_{2,\infty} \\
&C_1 + C_2 = 0
\end{aligned}$$

These relationships allow us to write (4.2.3) in terms of the exponents.

$$\begin{aligned}
y''(z) + &\left[\frac{1 - \alpha_{1,1} - \alpha_{2,1}}{z - z_1} + \frac{1 - \alpha_{1,2} - \alpha_{2,2}}{z - z_2}\right] y'(z) + \\
&\left[\frac{\alpha_{1,1}\alpha_{2,1}}{(z - z_1)^2} + \frac{\alpha_{1,2}\alpha_{2,2}}{(z - z_2)^2} + \frac{\alpha_{1,\infty}\alpha_{2,\infty} - \alpha_{1,1}\alpha_{2,1} - \alpha_{1,2}\alpha_{2,2}}{(z - z_1)(z - z_2)}\right] y(z) = 0. \qquad (4.2.4)
\end{aligned}$$

4.3. The Riemann-Papperitz Equation

Now assume (4.2.1) has three regular singular points, all finite, and that ∞ is an ordinary point. Denote the singularities by a, b, and c and denote the corresponding exponents by a' and a'', b' and b'', and c' and c''. Equation (4.2.1) then has the form

$$y''(z) + \frac{p_2(z)}{(z-a)(z-b)(z-c)} y'(z) + \frac{q_2(z)}{(z-a)^2(z-b)^2(z-c)^2} y(z) = 0. \qquad (4.3.1)$$

Exercise 4.3.1. *Use the fact that ∞ is an ordinary point of (4.3.1) to show that: (i) p_2 is a polynomial of degree two with the coefficient of z^2 equal to 2; (ii) q_2 is a polynomial of degree ≤ 2.*

From Exercise 4.3.1, we see that there exist constants A_1, A_2, A_3, B_1, B_2, and B_3 such that

$$\frac{p_2(z)}{(z-a)(z-b)(z-c)} = \frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c},$$
$$\frac{q_2(z)}{(z-a)^2(z-b)^2(z-c)^2} = \frac{B_1}{(z-a)^2} + \frac{B_2}{(z-b)^2} + \frac{B_3}{(z-c)^2},$$

and $A_1 + A_2 + A_3 = 2$. The form of the DE is now

$$y''(z) + \left[\frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c}\right] y'(z) + \left[\frac{B_1}{(z-a)^2} + \frac{B_2}{(z-b)^2} + \frac{B_3}{(z-c)^2}\right] \frac{y(z)}{(z-a)(z-b)(z-c)} = 0. \tag{4.3.2}$$

Exercise 4.3.2. *Using the indicial equations for (4.3.2), show that*

$$\begin{aligned}
a' + a'' &= 1 - A_1 \\
b' + b'' &= 1 - A_2 \\
c' + c'' &= 1 - A_3 \\
a'a'' &= \frac{B_1}{(a-b)(a-c)} \\
b'b'' &= \frac{B_2}{(b-a)(b-c)} \\
c'c'' &= \frac{B_3}{(c-a)(c-b)} \\
a' + a'' + b' + b'' + c' + c'' &= 1
\end{aligned}$$

So, in terms of the exponents, (4.3.2) becomes

$$\begin{aligned}
y''(z) + &\left[\frac{1-a'-a''}{z-a} + \frac{1-b'-b''}{z-b} + \frac{1-c'-c''}{z-c}\right] y'(z) + \\
&\left[\frac{a'a''(a-b)(a-c)}{z-a} + \frac{b'b''(b-a)(b-c)}{z-b} + \frac{c'c''(c-a)(c-b)}{z-c}\right] \frac{y(z)}{(z-a)(z-b)(z-c)} = 0
\end{aligned} \tag{4.3.3}$$

This is the *Riemann-Papperitz* equation. If y is a solution of the Riemann-Papperitz equation, we use the *Riemann P-function* notation

$$y = P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix}.$$

The right side is simply a symbol used to explicitly exhibit the singularities and their exponents. If c is replaced by ∞ then y satisfies the DE with $c \to \infty$. It can be shown that the results agree with what we got in the last section.

There are two useful properties of the Riemann P-function we will need later.

Theorem 4.3.1. *If a linear fractional transformation of the form*

$$z = \frac{At+B}{Ct+D} = \frac{(a-b)(a-c)(c_1-b_1)(t-a_1)}{(a-b)(a_1-c_1)(t-b_1) + (a-c)(b_1-a_1)(t-c_1)} + a$$

transforms a, b, and c into a_1, b_1, and c_1 respectively, then

$$P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = P\begin{pmatrix} a_1 & b_1 & c_1 & \\ a' & b' & c' & t \\ a'' & b'' & c'' & \end{pmatrix}.$$

This can be verified by direct, but tedious, substitution.

Theorem 4.3.2.

$$P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)^k P\begin{pmatrix} a & b & c & \\ a'-k & b'+k & c' & z \\ a''-k & b''+k & c'' & \end{pmatrix}.$$

Outline of Proof. If $w(z)$ satisfies (4.3.3), let $w(z) = \left(\frac{z-a}{z-b}\right)^k w_1(z)$. We will show that w_1 satisfies an equation of the form (4.3.3), but with the exponent a' replaced by $a'-k$. Corresponding to the regular singular point $z = a$, there is a Frobenius series solution corresponding to the exponent a':

$$w = \sum_{n=0}^{\infty} a_n (z-a)^{n+a'}.$$

Thus,

$$w_1(z) = (z-b)^k \sum_{n=0}^{\infty} a_n (z-a)^{n+a'-k}.$$

But $(z-b)^k$ is analytic at $z = a$, and has a series expansion around $z = a$

$$(z-b)^k = (a-b)^k + \sum_{n=1}^{\infty} b_n (z-a)^n,$$

so we can write

$$w_1(z) = \sum_{n=0}^{\infty} c_n (z-a)^{n+a'-k}$$

where $c_0 \neq 0$. Thus, the a' in the symbol for the Riemann P-function for w becomes $a'-k$ in the symbol for w_1. The other three exponents are similar. ♠

Exercise 4.3.3. *The transformation $y(z) = f(z)\,v(z)$ applied to (4.2.1) yields*

$$v'' + \left(2\frac{f'}{f} + p\right)v' + \left(\frac{f''}{f} + p\frac{f'}{f} + q\right)v = 0.$$

Also, $\frac{f'}{f} = (\log f)'$ and $\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2$. Now apply $w(z) = \left(\frac{z-a}{z-b}\right)^k w_1(z)$ to (4.3.3). Show that the indicial equation at $z = a$ is transformed from

$$r^2 - (a' + a'')\,r + a'a'' = 0$$

into

$$r^2 - (a' + a'' - 2k)\,r + k^2 - (a' + a'')\,k + a'a'' = 0.$$

Based on this work, prove Theorem 4.3.2.

4.4. The Hypergeometric Equation

Theorems 4.3.1 and 4.3.2 can be used to reduce (4.3.3) to a simple canonical form. Let $w(z)$ be a solution of (4.3.3) as before and let $w(z) = \left(\frac{z-a}{z-b}\right)^{a'} w_1(z)$. Then by Theorem 4.3.2, w_1 is a solution of the DE corresponding to

$$P\begin{pmatrix} a & b & c & \\ 0 & b'+a' & c' & z \\ a''-a' & b''+a' & c'' & \end{pmatrix}$$

Another zero exponent can be obtained by letting $w_1(z) = \left(\dfrac{z-b}{z-c}\right)^{b'+a'} w_2(z)$ so that w_2 is represented by

$$P \begin{pmatrix} a & b & c & \\ 0 & 0 & c'+b'+a' & z \\ a''-a' & b''-b' & c''+b'+a' & \end{pmatrix}$$

Note that the sum of the six exponents is still 1. Now let $\alpha = a'+b'+c'$, $\beta = a'+b'+c''$, and $\gamma = 1-a''+a'$. The Riemann P-function representing w_2 is now

$$P \begin{pmatrix} a & b & c & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}.$$

Penultimately, use a linear fractional transformation to map a, b, and c to 0, 1, and ∞ respectively:

$$t = \frac{(b-c)(z-a)}{(b-a)(z-c)}.$$

Finally, rename t to be z. We have the Riemann P-function $P \begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}$, which corresponds to the *hypergeometric DE*:

$$z(1-z)y''(z) + [\gamma - (\alpha+\beta+1)z]\, y'(z) - \alpha\,\beta\, y(z) = 0 \tag{4.4.1}$$

Exercise 4.4.1. *Fill in the details in the derivation of (4.4.1).*

Since (4.4.1) has a regular singular point at $z = 0$ with one exponent 0, one solution has the form

$$y = \sum_{k=0}^{\infty} a_k\, z^k$$

and the usual series manipulations lead to the recurrence relation

$$a_k = \frac{(k-1+\alpha)(k-1+\beta)}{k(k-1+\gamma)}\, a_{k-1}.$$

If we set $a_0 = 1$, we get the *hypergeometric function* $F(\alpha, \beta; \gamma; z)$ as a solution.

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k\, k!} z^k,$$

provided $\gamma \neq 0, -1, -2, \ldots$. If we also assume $\gamma \neq 1, 2, 3, \ldots$ the solution around $z = 0$ corresponding to the other exponent, $1-\gamma$, is

$$y_2(z) = \sum_{k=0}^{\infty} \frac{(1-\gamma+\alpha)_k (1-\gamma+\beta)_k}{(2-\gamma)_k\, k!} z^{k+1-\gamma} = z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; z)$$

Many known functions can be expressed in terms of the hypergeometric function. Here are some examples.

Example 4.4.1. *Polynomials. If either α or β is zero or a negative integer the series terminates.*

$$F(\alpha, 0; \gamma; z) = 1, \qquad F(\alpha, -n; \gamma; z) = \sum_{k=0}^{n} \frac{(\alpha)_k (-n)_k}{(\gamma)_k \, k!} \, z^k.$$

Example 4.4.2. *Logarithms.* $z \, F(1, 1; 2; -z) = \log(1+z)$ *and* $2z \, F(\frac{1}{2}, 1; \frac{3}{2}; z^2) = \log \frac{1+z}{1-z}$.

Example 4.4.3. *Inverse trigonometric functions.*

$$z \, F(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; z^2) = \arcsin z \quad \text{and} \quad z \, F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2) = \arctan z.$$

Example 4.4.4. *Rational functions and/or binomial expansions.* $F(\alpha, \beta; \beta; z) = \frac{1}{(1-z)^\alpha} = (1-z)^{-\alpha}$.

Example 4.4.5. *Complete elliptic integrals. In the following, z is the modulus, not the parameter.*

$$K(z) = \tfrac{\pi}{2} \, F(\tfrac{1}{2}, \tfrac{1}{2}; 1; z^2), \quad \text{and} \quad E(z) = \tfrac{\pi}{2} \, F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; z^2).$$

(See Math Mag. 68(3), June 1995, p.216 for an article on the rate of convergence of these hypergeometric functions.)

Exercises 4.4.2-6. *Verify the claims in Examples 4.4.1-5.*

Example 4.4.6. *Legendre polynomials. For n a positive integer, $P_n(z) = F(-n, n+1; 1; \frac{1-z}{2})$. This can be seen from the form of the series solution (see Example 4.1.1)), or can be derived directly from Legendre's DE, $(1-z^2)y''(z) - 2z \, y'(z) + n(n+1) \, y(z) = 0$, n a positive integer. The regular singular points are at ± 1 and ∞, and the transformation $t = \frac{1-z}{2}$ takes $1 \to 0$, $-1 \to 1$, and $\infty \to \infty$. The DE becomes*

$$t(1-t)y''(t) + (1-2t)y'(t) + n(n+1)y(t) = 0,$$

which can be seen to be the hypergeometric DE in t with $\alpha = -n$, $\beta = n+1$, and $\gamma = 1$.

Exercise 4.4.7. $F(\alpha, \beta; \gamma; z) = F(\beta, \alpha; \gamma; z)$.

Exercise 4.4.8. $\dfrac{d}{dz} \, F(\alpha, \beta; \gamma; z) = \dfrac{\alpha \, \beta}{\gamma} \, F(\alpha+1, \beta+1; \gamma+1; z)$.

Hypergeometric functions in which α, β, or γ are replaced by $\alpha \pm 1$, $\beta \pm 1$, or $\gamma \pm 1$ are called *contiguous* to $F(\alpha, \beta; \gamma; z)$. Gauss proved that $F(\alpha, \beta; \gamma; z)$ and any two of its contiguous functions are related by a linear relation with coefficients linear functions of z. The following exercises illustrate two such relations. There are many more.

Exercise 4.4.9. $(\gamma - \alpha - \beta) \, F(\alpha, \beta; \gamma; z) + \alpha(1-z) \, F(\alpha+1, \beta; \gamma; z) - (\gamma - \beta) \, F(\alpha, \beta-1; \gamma; z)$.

Exercise 4.4.10. $F(\alpha, \beta+1; \gamma; z) - F(\alpha, \beta; \gamma; z) = \frac{\alpha \, z}{\gamma} F(\alpha+1, \beta+1; \gamma+1; z) = 0$.

The infinite product result in Section 2.5 can be used to evaluate $F(\alpha, \beta; \gamma; 1)$ in terms of gamma functions. Details can be found in Whittaker and Watson, pp. 281-2. Limits are necessary because $z = 1$ is a singular point of the hypergeometric differential equation.

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \, \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \, \Gamma(\gamma - \beta)}.$$

4.5. Confluence of Singularities

Many differential equations of interest have an irregular singular point. The harmonic oscillator equation, $y'' + y = 0$, has an irregular singularity at ∞, for example. The results for DEs of Fuchsian type can be used to study such equations under the right circumstances. We may let two singularities come together and become an irregular singular point provided: (1) at least one of the corresponding exponents approaches ∞ and (2) the DE has a limiting form. This process, when posible, is called *confluence*. In this section we describe a general method to transform a Fuchsian DE by confluence into a DE with an irregular singularity.

Suppose we have a Fuchsian DE with singularities at 0, c, and ∞ and that the exponents at $z = c$ and $z = \infty$ depend on c. In order for the DE to have a limiting form as $c \to \infty$, it is necessary to require that the exponents at c and ∞ are linear functions of c. This will be assumed without proof. Thus, we can represent a solution of the DE by the Weierstrass P-function

$$P\begin{pmatrix} 0 & c & \infty & \\ \alpha_{1,1} & \alpha_{1,2} + c\,\beta_{1,2} & \alpha_{1,\infty} + c\,\beta_{1,\infty} & z \\ \alpha_{2,1} & \alpha_{2,2} + c\,\beta_{2,2} & \alpha_{2,\infty} + c\,\beta_{2,\infty} & \end{pmatrix}.$$

From (4.2.4) we get

$$\begin{aligned} y''(z) + & \left[\frac{1-\alpha_{1,1}-\alpha_{2,1}}{z} + \frac{1-\alpha_{1,2}-\alpha_{2,2}-c(\beta_{1,2}+\beta_{2,2}}{z-c}\right] y'(z) + \\ & \left[\frac{\alpha_{1,1}\alpha_{2,1}}{z^2} + \frac{\alpha_{1,2}\alpha_{2,2} + c(\alpha_{1,2}\beta_{2,2}+\alpha_{2,2}\beta_{1,2}) + c^2\beta_{1,2}\beta_{2,2}}{(z-c)^2}\right] y(z) + \\ & \left[\frac{\alpha_{1,\infty}\alpha_{2,\infty} - \alpha_{1,1,}\alpha_{2,1} - \alpha_{1,2}\alpha_{2,2}}{z(z-c)}\right] y(z) + \\ & \left[\frac{c(\alpha_{1,\infty}\beta_{2,\infty}+\alpha_{2,\infty}\beta_{1,\infty}-\alpha_{1,2}\beta_{2,2}-\alpha_{2,2}\beta_{1,2}) + c^2(\beta_{1,\infty}\beta_{2,\infty}-\beta_{1,2}\beta_{2,2})}{z(z-c)}\right] y(z) = 0. \end{aligned} \tag{4.5.1}$$

If a limiting form is to exist, we must have $\beta_{1,\infty}\beta_{2,\infty} - \beta_{1,2}\beta_{2,2} = 0$ to avoid the last term in (4.5.1) blowing up. The Fuchsian invariant has value 1, so

$$\alpha_{1,1} + \alpha_{2,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} + c(\beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty}) = 1.$$

This equation must hold for all c and for αs and βs independent of c, so

$$\begin{aligned} \beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty} &= 0 \\ \alpha_{1,1} + \alpha_{2,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} &= 1. \end{aligned}$$

Now let $c \to \infty$, giving

$$\begin{aligned} y''(z) + & \left[\frac{1-\alpha_{1,1}-\alpha_{2,1}}{z} + \beta_{1,2} + \beta_{2,2}\right] y'(z) + \\ & \left[\frac{\alpha_{1,1}\alpha_{2,1}}{z^2} + \beta_{1,2}\beta_{2,2} - \frac{\alpha_{1,\infty}\beta_{2,\infty}+\alpha_{2,\infty}\beta_{1,\infty}-\alpha_{1,2}\beta_{2,2}-\alpha_{2,2}\beta_{1,2}}{z}\right] y(z) = 0. \end{aligned} \tag{4.5.1}$$

Exercise 4.5.1. *Verify that equation (4.5.1) has an irregular singularity at ∞.*

Example 4.5.1. *Confluence can be used to obtain Bessel's DE of order n. This DE has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. The exponents at 0 are $\pm n$, so $\alpha_{1,1} = n$ and $\alpha_{2,1} = -n$, making (4.5.1)*

$$\begin{aligned} y''(z) + & \left[\frac{1}{z} + \beta_{1,2} + \beta_{2,2}\right] y'(z) + \\ & \left[\frac{-n^2}{z^2} + \beta_{1,2}\beta_{2,2} - \frac{\alpha_{1,\infty}\beta_{2,\infty}+\alpha_{2,\infty}\beta_{1,\infty}-\alpha_{1,2}\beta_{2,2}-\alpha_{2,2}\beta_{1,2}}{z}\right] y(z) = 0. \end{aligned}$$

We want to get Bessel's DE,

$$y''(z) + \frac{1}{z}\,y'(z) + \left(1 - \frac{n^2}{z^2}\right)\,y(z) = 0,$$

so, including the condition for the existence of a limiting form and the Fuchsian invariant, we get the following system of equations for the parameters.

$$\begin{gathered}
\beta_{1,2} + \beta_{2,2} = 0 \\
\beta_{1,2}\beta_{2,2} = 1 \\
\alpha_{1,\infty}\beta_{2,\infty} + \alpha_{2,\infty}\beta_{1,\infty} - \alpha_{1,2}\beta_{2,2} - \alpha_{2,2}\beta_{1,2} = 0 \\
\beta_{1,\infty}\beta_{2,infty} - \beta_{1,2}\beta_{2,2} = 0 \\
\beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty} = 0 \\
\alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} = 1
\end{gathered}$$

Since there are six equations in eight unknowns, any solution will contain two undetermined parameters. One such solution is

$$\beta_{1,2} = i, \quad \beta_{2,2} = -i, \quad \beta_{1,\infty} = i, \quad \beta_{2,\infty} = -i, \quad \alpha_{1,\infty} = \tfrac{1}{2} - \alpha_{2,2}, \quad \alpha_{2,\infty} = \tfrac{1}{2} - \alpha_{1,2}.$$

This shows that if we let $c \to \infty$ in the DE defined by

$$P\begin{pmatrix} 0 & c & \infty & \\ n & \alpha_{1,2} + ic & \frac{1}{2} - \alpha_{2,2} + ic & z \\ -n & \alpha_{2,2} - ic & \frac{1}{2} - \alpha_{1,2} - ic & \end{pmatrix},$$

the result is Bessel's DE.

Exercise 4.5.2. *Fill in the details in Example 4.5.1.*

The point of this section is that by means of the process of confluence, known results about the Fuchsian DE can suggest new results or avenues of study for the DE with an irregular singularity. The example involving Bessel's equation is to be taken as an illustration of the process. Actually, much more is known about Bessel's DE than the Fuchsian DE.

Exercise 4.5.3. *Show that the confluent equation obtained by letting $c \to \infty$ in the DE defined by*

$$P\begin{pmatrix} 0 & c & \infty & \\ \frac{1}{2} + m & c - k & -c & z \\ \frac{1}{2} - m & k & 0 & \end{pmatrix}$$

is $\dfrac{d^2u}{dz^2} + \dfrac{du}{dz} + \left(\dfrac{k}{z} + \dfrac{\frac{1}{4} - m^2}{z^2}\right) u = 0$. *Then let $u = e^{-z/2}W_{k,m}(z)$ to get Whittaker's equation for $W_{k,m}$:*

$$\frac{d^2W}{dz^2} + \left[-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right] W = 0.$$

Verify that Whittaker's equation has a regular singular point at 0 and an irregular singular point at ∞.

4.6. Generalized Hypergeometric Functions

A little time spent studying the series form of the basic hypergeometric function,

$$F(\alpha,\beta;\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!},$$

will suggest the question, "Why be restricted to just α, β, and γ for the factorial functions? Why not allow an arbitrary number of factorials in both numerator and denominator?" (OK, that's two questions, but, as you may know, there are three kinds of mathematicians: those who can count and those who cannot.) Thus we are led to consider the *generalized hypergeometric functions*, denoted ${}_pF_q(\alpha_1,\dots,\alpha_p;\gamma_1,\dots,\gamma_q;z)$ and defined by

$${}_pF_q(\alpha_1,\dots,\alpha_p;\gamma_1,\dots,\gamma_q;z)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}.$$

Exercise 4.6.1. *If $p\le q$, the series for ${}_pF_q$ converges for all z.*

Exercise 4.6.2. *If $p=q+1$, the series converges for $|z|<1$ and, unless it terminates, diverges for $|z|\ge 1$.*

Exercise 4.6.3. *If $p>q+1$, the series, unless it terminates, diverges for $z\ne 0$.*

Either p or q or both may be zero, and if this occurs, the absence of parameters will be denoted by a dash, $-$, in the appropriate position.

Example 4.6.1. ${}_0F_0(-;-;z)=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$

Example 4.6.2. ${}_1F_0(\alpha;-;z)=F(\alpha,\beta;\beta;z)=(1-z)^{-\alpha}.$

Example 4.6.3. ${}_0F_1(-;\gamma;z)=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{(\gamma)_k k!}$, *and from this we get* $J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(-;\nu+1;-\frac{z^2}{4})$.

Since the hypergeometric function came from the hypergeometric DE, the generalized hypergeometric functions should also satisfy appropriate DEs. Let the differential operator $\theta=z\frac{d}{dz}$. In terms of θ, the hypergeometric DE is

$$[\theta(\theta+\gamma-1)-z(\theta+\alpha)(\theta+\beta)]\;y=0.$$

Now if

$$y(z)={}_pF_q(z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\gamma_1)_k\cdots(\gamma_q)_k}\frac{z^k}{k!},$$

and since $\theta\,z^k=k\,z^k$, we get

$$\theta\prod_{j=1}^{q}(\theta+\gamma_j-1)\,y=\sum_{k=1}^{\infty}\frac{k\prod_{j=1}^{q}(k+\gamma_j-1)\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}=\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_{k-1}}\frac{z^k}{(k-1)!}.$$

Shifting the index gives

$$\theta\prod_{j=1}^{q}(\theta+\gamma_j-1)\,y=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_{k+1}}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^{k+1}}{k!}=z\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i+k)\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}=z\prod_{i=1}^{p}(\theta+\alpha_i)\,y.$$

Thus, if $p \le q+1$, we see that $y = {}_pF_q$ satisfies

$$\left[\theta \prod_{j=1}^{q}(\theta+\gamma_j-1) - z\prod_{i=1}^{p}(\theta+\alpha_i)\right] y = 0.$$

This DE is of order $q+1$ and, if no γ_j is a positive integer and no two γ_j's differ by an integer, the general solution is

$$y = \sum_{m=0}^{q} c_m\, y_m$$

where, for $m = 1, 2, \ldots, q$,

$$\begin{aligned}
y_0 &= {}_pF_q(\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_q;z)\\
y_m &= z^{1-\gamma_m}\,{}_pF_q(\alpha_1-\gamma_m+1,\ldots,\alpha_p-\gamma_m+1;\\
&\quad \gamma_1-\gamma_m+1,\ldots,\gamma_{m-1}-\gamma_m+1, 2-\gamma_m, \gamma_{m+1}-\gamma_m+1,\ldots,\gamma_q-\gamma_m+1;z)
\end{aligned}$$

Exercise 4.6.4. *Find the DE satisfied by* ${}_3F_2(2,2,2;\frac{5}{2},4;z)$. *Also find the general solution of this DE.*

Exercise 4.6.5. *Show that if* y_1 *and* y_2 *are linearly independent solutions of*

$$y''(z) + p(z)\,y'(z) + q(z)\,y(z) = 0$$

then three linearly independent solutions of

$$w'''(z) + 3p(z)\,w''(z) + \left(2p^2(z) + p'(z) + 4q(z)\right)\,w'(z) + \left(4p(z)q(z) + 2q'(z)\right)\,w(z) = 0$$

are $y_1^2(z)$, $y_1(z)y_2(z)$, *and* $y_2^2(z)$.

Exercise 4.6.6. *Show that* ${}_3F_2(2,2,2;\frac{5}{2},4;z) = \left(F(1,1;\frac{5}{2};z)\right)^2$. *[Hint: Use Exercises 4.6.4 and 4.6.5.]*

Chapter 5. Orthogonal Functions

5.1. Generating Functions

Consider a function f of two variables, (x, t), and its formal power series expansion in the variable t:

$$f(x,t) = \sum_{k=0}^{\infty} g_k(x)\, t^k.$$

The coefficients in this series are, in general, functions of x, and we can think of them as having been "generated" by the function f. In fact, $g_k(x) = \dfrac{1}{k!}\dfrac{\partial^k f}{\partial t^k}(x,0)$, though there may be better ways to compute them. If this idea is extended slightly, we get the following definition:

Definition 5.1.1. *The function $F(x,t)$ is a generating function for the sequence $\{g_k(x)\}$ if there exists a sequence of constants $\{c_k\}$ such that*

$$F(x,t) = \sum_{k=0}^{\infty} c_k g_k(x)\, t^k.$$

It is not uncommon for all the c_ks to be one. One of the principal problems involving generating functions is determining a generating function for a given set or sequence of polynomials. Especially desirable is a general theory which can be used to get generating functions. Unfortunately, no such theory has yet been developed, so we must be content with results for special cases found using manipulative dexterity.

Example 5.1.1. *One special case is when the coefficients are successive powers of the same function. Let $\{g_k(x)\} = \{(f(x))^k\}$. Then the generating function can be found using the formula for the sum of a geometric series.*

$$F(x,t) = \sum_{k=0}^{\infty} (f(x))^k\, t^k = \frac{1}{1 - t\, f(x)}$$

provided $|f(x)| < 1$.

Exercise 5.1.1. *Find the generating function for the sequence $\{k\,(f(x))^k\}$.*

Many sets of elementary and special functions have known generating functions. Here are some examples.

Example 5.1.2. *The Bernoulli functions. Let $y(x,t) = \sum_{k=0}^{\infty} B_k(x)\, t^k$. Termwise differentiation with respect to x and properties of the Bernoulli functions (section 1.3) yields $y_x(x,t) = t\, y(x,t)$. Thus,*

$$\frac{\partial}{\partial x}\left[e^{-xt} y(x,t)\right] = e^{-xt}\left[y_x(x,t) - t\, y(x,t)\right] = 0,$$

and so for each t there is a function $C(t)$ such that $y(x,t) = C(t)e^{xt}$, and we have

$$C(t)e^{xt} = \sum_{k=0}^{\infty} B_k(x)\, t^k. \tag{5.1.1}$$

Integration of (5.1.1) over the interval $[0, 1]$ and properties of the Bernoulli functions give

$$C(t) = \frac{t}{e^t - 1}.$$

Thus, the generating function for the Bernoulli functions is $y(x,t) = \dfrac{te^{xt}}{e^t - 1}$.

Exercise 5.1.2. *Fill in the details in Example 5.1.2.*

Example 5.1.3. *Legendre polynomials:* $(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(x) t^k$.

Exercise 5.1.3. *Use Taylor's theorem to verify the first three coefficients in the generating function relation for the Legendre polynomials.*

Example 5.1.4. *Bessel functions:* $\exp\left[\frac{1}{2} z(t - \frac{1}{t})\right] = \sum_{k=-\infty}^{\infty} J_k(z) t^k$.

Example 5.1.5. *Hermite polynomials. Denote the Hermite polynomial of degree n by $H_n(x)$. Then* $\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \dfrac{H_n(x)}{n!}\, t^n$.

Exercise 5.1.4. *Find the first four Hermite polynomials.*

Exercise 5.1.5. *Prove the expansions*

$$e^{t^2} \cos 2xt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x)}{(2n)!}\, t^{2n}$$

$$e^{t^2} \sin 2xt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}(x)}{(2n+1)!}\, t^{2n+1}$$

for $|t| < \infty$. These expressions can be thought of as generating functions for the even and odd Hermite polynomials.

The generating functions for both the Legendre and the Hermite polynomials are functions of the form $G(2xt - t^2)$. The following theorem is representative of theorems which give properties common to all sets of functions having generating functions of this form.

Theorem 5.1.1. *If $G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n$, then $g_0'(x) = 0$ and, for $n \geq 1$, the g_ns satisfy the differential-difference equation*

$$x\, g_n'(x) - n\, g_n(x) = g_{n-1}'(x).$$

Proof. Let $F(x,t) = G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n$. Then F satisfies the PDE $(x - t)\dfrac{\partial F}{\partial x} - t\dfrac{\partial F}{\partial t} = 0$. In terms of the series, this PDE is

$$\sum_{n=0}^{\infty} x\, g_n'(x)\, t^n - \sum_{n=0}^{\infty} n\, g_n(x)\, t^n = \sum_{n=1}^{\infty} g_{n-1}'(x)\, t^n.$$

Equating coefficients gives the desired result. ♠

Exercise 5.1.6. *In A&S, pages 783-4, a number of generating functions are given as functions of* $R = \sqrt{1 - 2xt + t^2}$. *Formulate and prove the equivalent of Theorem 5.1.1 using* R *in place of* $2xt - t^2$.

5.2. Orthogonality

Consider the DE
$$a_0(x)\, y'' + a_1(x)\, y' + [a_2(x) + \lambda]\, y = 0.$$

Multiply by the "integrtating factor" $p(x) = \exp \frac{a_1(x)}{a_0(x)}\, dx$, let $q(x) = \frac{a_2(x)}{a_0(x)}\, p(x)$, and $r(x) = \frac{p(x)}{a_0(x)}$, to get the DE into the form
$$[p(x)\, y']' + [q(x) + \lambda\, r(x)]\ y = 0. \tag{5.2.1}$$

Equation (5.2.1) is said to be in *Sturm-Liouville form* and if appropriate boundary conditions are specified on an interval we have a *Sturm-Liouville problem.* Values of λ for which a Sturm-Liouville problem (SLP) has nontrivial solutions are called *eigenvalues* of the SLP and the corresponding solutions are called *eigenfunctions.* These ideas are studied in detail in courses on partial differential equations and boundary value problems where the SLP arises naturally in the solution of PDEs with boundary conditions. The following theorem, stated here rather vaguely, is proved in such courses.

Theorem 5.2.1. *Under appropriate conditions, if* y_m *and* y_n *are eigenfunctions corresponding to distinct eigenvalues of the SLP associated with (5.2.1) on the interval* $[a, b]$, *then*

$$\int_a^b r(x) y_m(x) y_n(x)\, dx = 0. \tag{5.2.2}$$

When equation (5.2.2) holds, we say that y_m and y_n are *orthogonal* with respect to the weight function $r(x)$. This equation can, in fact, be taken as the definition of orthogonality.

Exercise 5.2.1. *What do you call a tornado at the Kentucky Derby?*

Example 5.2.1 (Legendre). *Legendre's DE, as we have seen, is* $(1 - x^2)\, y'' - 2\, x\, y' + n(n + 1)\, y = 0$. *In Sturm-Liouville form, this becomes*

$$\left[(1 - x^2)\, y'\right]' + n(n + 1)\, y = 0.$$

Here, $p(x) = 1 - x^2$, $q(x) \equiv 0,$, $r(x) \equiv 1$, *and* $\lambda = n(n+1)$. *Since* $x = \pm 1$ *are regular singular points, we can be sure solutions exist on the closed interval* $[-1, 1]$ *only when the solutions are polynomials, so the eigenvalues are* $n = 0, 1, 2, \ldots$ *and the eigenfunctions are the corresponding Legendre polynomials* $P_0(x), P_1(x), P_2(x), \ldots$. *By Theorem 5.2.1 we have*
$$\int_{-1}^1 P_m(x)\, P_n(x)\, dx = 0$$
whenever m *and* n *are distinct nonnegative integers.*

The orthogonality integral is a generalization to functions of the dot product for vectors, and since the dot product of a vector with itself is the square of the length of the vector, the integral in (5.2.2) with both eigenfunctions the same can be interpreted as the "length" squared of the eigenfunction. Often, we want this length to be one for all the eigenfunctions, in which case we say that the eigenfunctions are *normalized.* Since the eigenfunctions are orthogonal by (5.2.2), if they are also normalized, we say they are *orthonormal.*

Example 5.2.1 (continued). *We now determine* $\int_{-1}^{1} P_n^2(x)\,dx$ *using the generating function.*

$$\begin{aligned}
(1-2xt+t^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} P_n(x)\,t^n \\
(1-2xt+t^2)^{-1} &= \left[\sum_{n=0}^{\infty} P_n(x)\,t^n\right]^2 \\
\frac{1}{t}\log\left|\frac{1+t}{1-t}\right| &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx \\
2\left(1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots+\frac{t^{2n}}{2n+1}+\cdots\right) &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx.
\end{aligned}$$

Equating coefficients gives the normalizing constants for the Legendre polynomials:

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Exercise 5.2.2. *Fill in the details in Example 5.2.1.*

Example 5.2.2 (Bessel). *Bessel's DE of order* n*, (slightly modified - do you see how?)* $x^2y'' + x\,y' + (\lambda^2x^2 - n^2)\,y = 0$*, written in Sturm-Liouville form is*

$$[x\,y']' + \left[\lambda^2 x - \frac{n^2}{x}\right]\,y = 0.$$

For the interval $[0, b]$*, the eigenvalues are* $\lambda_k = \frac{\alpha_k}{b}$*, where* α_k *is the* k^{th} *positive zero of* $J_n(x)$*. The orthogonality integral is, for* $m \neq k$*,*

$$\int_0^b x\,J_n(\lambda_m x)\,J_n(\lambda_k x)\,dx = 0.$$

Note here that n *is fixed, and the different eigenvalues and eigenfunctions are denoted by the subscripts on* λ *or* α*.*

For sets of polynomials, the following equivalent condition for orthogonality is often useful. We call a set of polynomials *simple* if the set contains exactly one polynomial of each degree; unless stated otherwise, the degree of a subscripted polynomial is equal to its subscript.

Theorem 5.2.2. *If* $\{\phi_n(x)\}$ *is a simple set of real polynomials and* $r(x) > 0$ *on an interval* (a, b)*, then* $\{\phi_n(x)\}$ *is an orthogonal set with respect to the weight function* $r(x)$ *if and only if for* $k = 0, 1, 2, \ldots, n-1$*,*

$$\int_a^b r(x)\,x^k\,\phi_n(x)\,dx = 0.$$

Outline of Proof. The proof is based first on the fact that any polynomial of degree $m < n$ can be written as a linear combination of powers of x from x^0 through x^m. Then the fact that x^k can be expressed as a linear combination of $\phi_0(x)$ through $\phi_k(x)$ is used. Details are left to the student. ♠

Exercise 5.2.3. *Prove Theorem 5.2.2.*

Exercise 5.2.4. *Prove that if* $\{\phi_n(x)\}$ *is a simple set of real polynomials and* $r(x) > 0$ *on an interval* (a, b)*, then for every polynomial* P *of degree less than* n*,* $\int_a^b r(x)\, \phi_n(x)\, P(x)\, dx = 0$*. Also prove that* $\int_a^b r(x)\, x^n\, \phi_n(x)\, dx \neq 0$.

The interesting part of a polynomial is near the zeros. After the last zero and before the first one, polynomials are rather boring - they either go up, up, up, or down, down, down.

Theorem 5.2.3. *If* $\{\phi_n(x)\}$ *is a simple set of real polynomials, orthogonal with respect to a weight function* $r(x) > 0$ *on an interval* (a, b)*, then, for each* n*, the zeros of* ϕ_n *are distinct and all lie in the interval* (a, b).

Proof. For $n > 0$, by Theorem 5.2.2 $\int_a^b r(x)\, \phi_n(x)\, dx = 0$, so the integrand must change sign at least once in (a, b), and since $r(x) > 0$, this means $\phi_n(x)$ changes sign in (a, b). Let $\{\alpha_k\}_{k=1}^s$ be the set of points where $\phi_n(x)$ changes sign in (a, b). These are the zeros of ϕ_n of odd multiplicity, and since the degree of ϕ_n is n, we know that $s \leq n$. Form the polynomial

$$P(x) = \prod_{k=1}^{s} (x - \alpha_k).$$

Assume $s < n$. Then by Exercise 5.2.4,

$$\int_a^b r(x)\, \phi_n(x)\, P(x)\, dx = 0.$$

But all the zeros of $\phi_n(x)\, P(x)$ are of even multiplicity, so $r(x)\, \phi_n(x)\, P(x)$ cannot change sign in (a, b). Hence, $s < n$ is not possible, and we must have $s = n$. This means that ϕ_n has n roots of odd multiplicity in (a, b). Since the degree of ϕ_n is n, each root is simple, and the theorem is proved. ♠

5.3. Series Expansions

An important application of orthogonal polynomials in physics and engineering is the expansion of a given function in a series of the polynomials. For a simple set of polynomials, the powers of x in the usual series representation are replaced by the polynomials of appropriate degree. Of course, the problem is to find the coefficients in such a series expansion, and this is where orthogonality becomes quite useful.

Example 5.3.1. *Let* f *be defined in the interval* $(-1, 1)$*, and expand* $f(x)$ *in a series of Legendre polynomials. In other words, we want to determine the coefficients in*

$$f(x) = \sum_{n=0}^{\infty} c_n\, P_n(x) \tag{5.3.1}$$

so that equality holds for $x \in (-1, 1)$*. Proceeding formally, we multiply both sides by* $P_m(x)$ *and integrate from* -1 *to* 1.

$$\int_{-1}^{1} f(x)\, P_m(x)\, dx = \sum_{n=0}^{\infty} c_n \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1} c_m,$$

which implies that, for $n = 0, 1, 2, \ldots$

$$c_n = (n + \frac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\, dx. \tag{5.3.2}$$

This procedure is neat, clean, and algorithmic, but we took some mathematical liberties which should at least be acknowledged. In particular, how did we know that $f(x)$ could be represented as in (5.3.1) in the first place, and also, was it legitimate to interchange the operations of integration and summation? Unless these points are cleared up, we have no guarantee, except faith, that (5.3.1) with coefficients given by (5.3.2) converges and has sum $f(x)$. Another concern is that even if we can be sure the procedure works for Legendre polynomials, will a similar procedure be valid for a different set of simple orthogonal polynomials? Fortunately, for a given set of orthogonal polynomials, there are conditions which do guarantee that equations (5.3.1) and (5.3.2) or their equivalents are valid. Unfortunately, the conditions are different for different sets of polynomials. Proofs get somewhat involved, and are omitted here, but interested readers may consult Lebedev or Whittaker and Watson.

Theorem 5.3.1. *If the real function f is piecewise smooth in the interval $(-1, 1)$ and if $\int_{-1}^{1} f^2(x)\, dx$ is finite, then the Legendre series (5.3.1) with coefficients given by (5.3.2) converges to $f(x)$ wherever f is continuous. If x_0 is a point of discontinuity, the series converges to the average of the right-hand and left-hand limits of $f(x)$ at x_0.*

Exercise 5.3.1. *Expand $f(x) = x^2$ in a series of Legendre polynomials.*

Exercise 5.3.2. *Expand $f(x) = \begin{cases} 0, & -1 \le x < \alpha \\ 1, & \alpha < x \le 1 \end{cases}$ in a series of Legendre polynomials, and verify the value at $x = \alpha$.*

Exercise 5.3.3. *Express $f(x) = \sqrt{\frac{1-x}{2}}$ in a series of Legendre polynomials. Calculate the coefficients by using the generating function.*

It is possible to derive all properties of a set of orthogonal polynomials by starting with only the generating function. The following series of exercises builds up some results about the Hermite polynomials defined in Example 5.1.5.

Exercise 5.3.4. *Show that the generating function $F(x, t)$ for the Hermite polynomials satisfies $\frac{\partial F}{\partial x} - 2t\, F = 0$, and so $H_n'(x) = 2n\, H_{n-1}(x)$. Similarly, show that $F(x, t)$ satisfies $\frac{\partial F}{\partial t} - 2(x - t)\, F = 0$, and so*

$$H_{n+1}(x) - 2x\, H_n(x) + 2n\, H_{n-1}(x) = 0. \tag{5.3.3}$$

Exercise 5.3.5. *Show that the Hermite polynomials satisfy the differential equation (Hermite's DE)*

$$y''(x) - 2x\, y'(x) + 2n\, y(x) = 0.$$

Write Hermite's DE in Sturm-Liouville form and determine the interval and the weight function for the orthogonality of the Hermite polynomials.

Exercise 5.3.6. *In this exercise, you will calculate $\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx$. Begin by replacing the index n in (5.3.3) by $n - 1$ and multiply by $H_n(x)$. Then from this equation subtract (5.3.3) multiplied by $H_{n-1}(x)$.*

Work with this result to obtain

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx = 2n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}^2(x)\, dx$$

for $n = 2, 3, \ldots$. *Repeated application of this reduction formula gives, for* $n = 2, 3, \ldots$,

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx = 2^n\, n!\, \sqrt{\pi}. \tag{5.3.4}$$

Finally, show by direct calculation that (5.3.4) also holds for $n = 0, 1$.

There is a result for Hermite polynomials corresponding to Theorem 5.3.1, in which the integral required to be finite is $\int_{-\infty}^{\infty} e^{-x^2} f^2(x)\, dx$.

Exercise 5.3.7. *Expand* $f(x) = sgn(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ *in a series of Hermite polynomials.*