This is a closed-book, closed-notes exam. The only items you are allowed to use are writing implements. Mark each sheet of paper you use with your name and clearly indicate the problem number.

The max number of points per question is indicated in square brackets after each question. The sum of the max points for all the questions is 55, but note that the max exam score will be capped at 50 (i.e., there are 5 bonus points, but you cant score more than 100%). Partial credit will be awarded, so show your work!

You have exactly 60 minutes to complete this exam. Keep your answers clear and concise while complete. Good luck!

1. You are building a highly reliable computer that is designed to still function even if some of its components fail. It has three CPUs, two network cards, and four hard drives, each of which can be 'functional' or 'failed'.

   (a) (3 points) How many possible states does this system have? List five of them.

   Solution: Each component (CPU #1, CPU #2, CPU #3, etc.) can be either functional or failed. So for each of 9 components you have two choices: either the component is up or it is down. Therefore, there are 2^9 possibilities here.

   (b) (4 points) Suppose the system requires at least one functioning CPU, one functioning network card, and two functioning hard drives to work. How many of the possible system states result in a working system?

   Solution: For the CPUs, there are 2^3 outcomes but only one (all CPUs failed) leads to a failed system. Likewise, for the network cards there are 2^2 outcomes but only one that causes the system to fail.

   For the hard drives, there are 2^4 outcomes, and $1 + \binom{4}{1} = 5$ that result in failure (either all drives fail or only one remains functioning).

   Therefore, the number of system states where the system is functional is given by

   $$(2^3 - 1) * (2^2 - 1) * (2^4 - 5) = 7 * 3 * 11 = 231$$

   (c) (4 points) Assume each system state is equally probable.[1] What is the probability that the system does not work?

   Solution: Using the subtraction rule and the answer from the previous part,

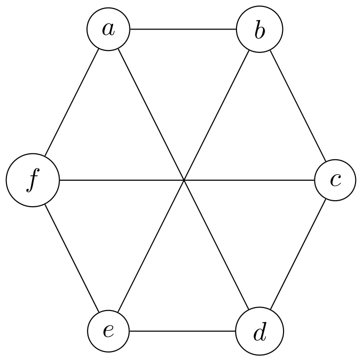
   $$1 - \frac{\text{\# of functional system states}}{2^9} = 1 - \frac{231}{512} = \frac{512 - 231}{512} = \frac{281}{512}$$

2. (3 points) In a group of six people is it possible for each person to have exactly three friends in the group? Explain why or why not.

   Solution: Yes! First, we can check that the total degree is OK: $6 * 3 = 18$ which is even so we are good.

   Next, let's draw a possible graph:

[1]Hopefully this isn't the case in reality!



3. (4 points) You write up a survey for a psychology class testing, uh... the effect of liking ice cream flavors on survey responses. Suppose 20 people take your survey, and the survey software tells you the following information:

Response	*Count*
Chocolate	10
Vanilla	15
Mint	11
Chocolate and Vanilla	7
Chocolate and Mint	8
Vanilla and Mint	6

(Each entry in the table counts the number of people who checked that box. Someone who checked 'chocolate' and 'vanilla' would be counted once for 'chocolate', once for 'vanilla', and once for 'chocolate and vanilla'.)

How many people like all three kinds of ice cream?

Solution: Let n be the number of people who like all three kinds of ice cream. Based on our knowledge about the number of elements in non-disjoint (overlapping) sets, we can determine the following:

$$20 = 10 + 15 + 11 - 7 - 8 - 6 + n$$

Solving for n, we get $n = 5$.

This can be checked by drawing a Venn diagram and determining the number of elements in each segment of the diagram.

4. A local sandwich shop offers seven kinds of ingredients (turkey, ham, roast beef, lettuce, tomatoes, onions, and pickles) and three condiments (mayo, mustard, and hot sauce).

   (a) (4 points) How many different sandwiches can you order with three unique ingredients and two unique condiments?

   Solution: We need to choose three distinct ingredientes and two distinct condiments. There are $\binom{7}{3}$ ways to pick ingredients and $\binom{3}{2}$ ways to pick condiments. Therefore, there are $\binom{7}{3} * \binom{3}{2}$ possible sandwich orders.

(Now, if you're of the opinion that it matters the way the ingredients and condiments are layered on the sandwich, there are 5! permutations of your choice of ingredients and condiments. Therefore, you'd have $\binom{7}{3} * \binom{3}{2} * 5!$ possible sandwich orders.)

(b) (4 points) How does your answer to part 1 change if you can repeat ingredients but not condiments?[2]

Solution: The ingredients are now chosen via repeated combinations. Here, we have $n = 7$ options and $r = 3$ choices. So, you have overall $\binom{7+3-1}{3} * \binom{3}{2}$ possible sandwich orders.

5. (6 points) You and your friends head over to the local pizza joint for a meeting of the Discreet Discrete Math Club.[3] If someone wishes to join the club, they must come to a meeting and tell the members a true fact derived from some mathematical knowledge.

While you are all engrossed in a hushed discussion of the finer points of binary trees, a stranger approaches the table and says, "At least three of you will order pizzas that share a topping." If there are seven of you at the table and the pizza joint offers three kinds of pizza topping, should you let the stranger join your club? Why or why not?

Solution: The generalized pigeonhole principle states that if you have n pigeons and k holes, then one hole will have more than

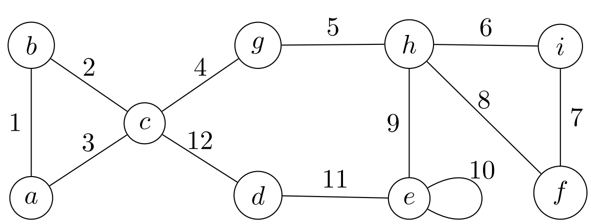
$$\left\lfloor \frac{n-1}{k} \right\rfloor$$

pigeons.

Here we have $n = 7$ pigeons (pizza orders) and $k = 3$ pigeonholes. So one pigeonhole will have more than $\frac{6}{3} = 2$ pigeons, implying that at least three people will order a pizza with the same topping. Therefore, said stranger is either very good at guessing or knows about the pigeonhole principle. Either way, they'll probably be a good club member!

(Technically, here we only know $n \geq 7$ since it is possible for someone to order a pizza with multiple toppings, but this just increases the number of pigeons and thus the minimum number of pigeons per hole, so our analysis is still sound.)

6. Look at this graph:



(a) (3 points) Write a path from b to h.

Solution: No need to get fancy here: $b2c4g5h$.

[2] Triple ham, that's fine; triple mayo. . . ew.

[3] The Discrete Discreet Math Club meetings are rather boring as everyone has to sit at their own table.

(b) (8 points) Find an Euler circuit for this graph following the algorithm shown in class.

Solution: Alright, let's start off by making a circuit:

$$C_1 = a1b2c3a$$

No repeated edges, which is good, but it doesn't cover the whole graph. So, let's make another circuit that touches this one at vertex c:

$$C_2 = c4g5h9e11d12c$$

Now we can patch them together:

$$C' = a1b2c4g5h9e11d12c3a$$

Another circuit touching this one:

$$C_3 = e10e$$

Small, but servicable. Patched into circuit C':

$$C'' = a1b2c4g5h9e10e11d12c3a$$

One circuit left:

$$C_4 = h6i7f8h$$

Adding this to C'' gives us a complete Euler circuit:

$$C = a1b2c4g5h6i7f8h9e10e11d12c3a$$

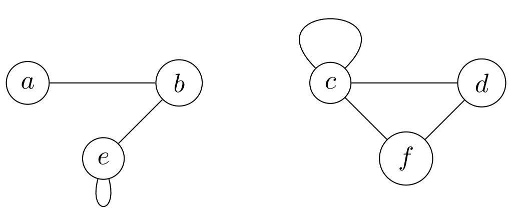
Note: It really doesn't matter what circuits you pick or the order you patch them together in; in fact you might be able to just write down the whole circuit by inspection. The solution presented here is one example of how this problem could be solved.

7. Let G be the graph given by the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) (4 points) Draw G.

Solution:



(b) (2 points) What is the total degree of G?

Solution: There are 7 edges, so the total degree is 14. Recall that a self-loop counts twice!

(c) (6 points) Is G connected? Why or why not? Your answer should incorporate what it means for a graph to be connected.

Solution:
No! To be connected, a walk must exist between any two vertices in the graph. As a counterexample, take vertices a and c. Walk around the graph as much as you like, but if you start from a you will never get to c.

This graph has two *connected components* – subgraphs that are connected.