Quantum Two
Many Particle Systems:
An Introduction to Direct Product Spaces
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Thus, the only real obstacle to our immediate application of the postulates to a system of many (possibly interacting) particles is that we have till now avoided the question of what the linear vector space, the state vector, and the operators of a many-particle quantum mechanical system look like.
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The state vector \( |\psi\rangle \) of a system of \( N \) particles is an element of the direct product space

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Clearly, to understand this principle we need to explore the structure of such **direct product spaces**.
The Direct Product of Quantum Mechanical State Spaces

Let $S_1$ and $S_2$ be the state spaces of two independent quantum mechanical systems, and let their dimensions be denoted by $N_1$ and $N_2$, respectively (either or both of which may be infinite).
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Each space might represent that of a single particle, or they may be more complicated spaces, but it is assumed that they each represent different independent quantum mechanical degrees of freedom.
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$|\phi\rangle^{(2)}$ represents a state of $S_2$. 
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To describe the **combined system** we now define a new state space

\[ S_{12} = S_1 \otimes S_2 \]

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That is, from each pair of states $|\psi\rangle^{(1)}$ in $S_1$ and $|\phi\rangle^{(2)}$ in $S_2$

we construct a direct product state of the direct product space $S_{12}$

$$|\psi, \phi\rangle \equiv |\psi\rangle^{(1)} \otimes |\phi\rangle^{(2)} = |\psi\rangle^{(1)} |\phi\rangle^{(2)}$$
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The direct product space $S_{12}$ for the combined system, then contains:

1. all such direct product states as well as (it follows automatically)
2. all possible linear combinations of those states (it has to be closed)
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This **direct product of states** is assumed to be **commutative** in a trivial sense that

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Thus, in the decoupled forms on the right we are free to move the two kets from each space past each other whenever it’s needed (and it often is).
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The direct product of states is also assumed to be **linearly distributive**

i.e., if

\[ |\psi\rangle^{(n)} = \alpha |\xi\rangle^{(n)} + \beta |\eta\rangle^{(n)}, \]

then

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is said to be an entangled state of the combined system.

In an entangled state neither subsystem can be described independently by its own state vector, without consideration of the state of the other.
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Generally, entangled states of a combined quantum system arise as a result of interactions between the independent degrees of freedom of each subsystem.

You may have heard that the existence and properties of entangled quantum mechanical states are essential elements in attempts to implement quantum computing and quantum cryptography.
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We have also introduced a mathematical definition of the direct product of quantum mechanical state spaces, which contain all direct products states, as well as all entangled states, formed from non-factorizeable linear combinations of direct product states.
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We have also introduced a mathematical definition of the direct product of quantum mechanical state spaces, which contain all direct products states, as well as all entangled states, formed from non-factorizeable linear combinations of direct product states.

In the next module we explore further some of the mathematical structure of direct products spaces, and learn a little bit more about how to actually compute useful things, e.g., inner products, how to construct appropriate operators, and how to generate sets of orthonormal basis vectors for direct product spaces.