CS128 Spr2012 Exam 2 KEY

This is a closed-book, closed-notes exam. The only items you are allowed to use are writing implements. Mark each sheet of paper you use with your name and the string "`cs128 Spr2012 exam#2`". If you are caught cheating, you will receive a zero grade for this exam. The number of points each question is worth is indicated in square brackets after each question. The sum of the points for all the questions is 55, but note that the exam's score will be capped at 50 (i.e., there are 5 bonus points but you can't score more than 50). You have exactly 50 minutes to complete this exam. Keep your answers clear and concise while complete. Reminder: $\mathbb{Z}^+$ is the set of all positive integers. Good luck!

1. Let $A = \{a, b, \{c, d\}\}$ and $B = \{\{a, b\}, \{c, d\}\}$. Answer "true" or "false":
   (a) Is $A \subseteq B$? *[1] false*
   (b) Is $B \subseteq A$? *[1] false*
   (c) Is A a proper subset of B? *[1] false*
   (d) Is B a proper subset of A? *[1] false*
   (e) Is $A = B$? *[1] false*
   (f) Is $A \subseteq \emptyset$? *[1] false*
   (g) Is $\emptyset \subseteq A$? *[1] true*
   (h) Is $\emptyset \in \emptyset$? *[1] false*

2. Let $A = \{a, b, \{c, d\}\}$ and $B = \{\{a, b\}, \{c, d\}\}$.
   (a) Give $A \cap B$? *[2]* $\{\{c, d\}\}$
   (b) Give $B \cup A$? *[2]* $\{a, b, \{c, d\}, \{a, b\}\}$

3. Let $S_i = \{x \in \mathbb{Z}^+ | i \text{ divides } x\}$. List at least 5 elements of:
   (a) $\bigcup_{i=3}^{5} S_i$. *[3]* { 3,4,5,6,8,9,10,12,15,16,...}
   (b) $\bigcap_{i=5}^{6} S_i$. *[3]* { 30,60,90,120,150,...,300,...}

4. Let $A = \{1, 2\}, B = \{a, b\}$ and $C = \{a, 1\}$
   (a) Give $A \times (B \cap C)$. *[3]*
   $B \cap C = \{a\}$
   $A \times (B \cap C) = \{1, 2\} \times \{a\} = \{(1, a), (2, a)\}$
   (b) Give $\mathcal{P}(A \times (B - C))$ where $\mathcal{P}$ is the power set. *[4]*
   $B - C = \{b\}$
   $A \times (B - C) = \{1, 2\} \times \{b\} = \{(1, b), (2, b)\}$
   $\mathcal{P}(\{(1, b), (2, b)\}) = \{\emptyset, \{(1, b)\}, \{(2, b)\}, \{(1, b), (2, b)\}\}$

5. Write $(1 - t) \cdot (2 - t^2) \cdot (3 - t^3) \cdot (4 - t^4) \cdot (5 - t^5)$ using summation or product notation. *[3]*
   $\prod_{k=1}^{5} (k - t^k)$

6. Compute $\binom{7}{5}$, simplify as much as possible. *[3]*
   $\binom{7}{5} = \frac{7!}{5!(7-5)!} = \frac{7!}{5!2!} = \frac{2\cdot3\cdot4\cdot5\cdot6\cdot7}{2\cdot3\cdot4\cdot5\cdot2} = \frac{6\cdot7}{2} = \frac{42}{2} = 21$

7. (Induction Proof) Let n be a positive integer and $P(n)$ be the formula

$$1 + 6 + 11 + 16 + \ldots + (5n - 4) = \frac{n(5n-3)}{2}$$

(a) *(Base Case)*. Show that $P(1)$ is true. *[4]*

$$\begin{aligned} 1 &= \frac{1(5(1)-3)}{2} \\ &= \frac{2}{2} \\ 1 &= 1 \end{aligned}$$

(b) *(Inductive Hypothesis)*. State $P(k)$. *[1]*

$$1 + 6 + 11 + 16 + \ldots + (5k - 4) = \frac{k(5k-3)}{2}$$

(c) *(Conclusion)*. State $P(k+1)$. *[1]*

$$1 + 6 + 11 + 16 + \ldots + (5k - 4) + (5(k+1) - 4) = \frac{(k+1)(5(k+1)-3)}{2}$$

Note:

$$\begin{aligned} \frac{(k+1)(5(k+1)-3)}{2} &= \frac{(k+1)(5k+5-3)}{2} \\ &= \frac{(k+1)(5k+2)}{2} \\ &= \frac{5k^2+2k+5k+2}{2} \\ &= \frac{5k^2+7k+2}{2} \end{aligned}$$

(d) *(Inductive Step)*. Show that $P(k) \to P(k+1)$. *[10]* By the induction hypothesis, let

$$1 + 6 + 11 + 16 + \ldots + (5k - 4) = \frac{k(5k-3)}{2}$$

Add $(5(k+1) - 4)$ to both sides

$$\begin{aligned} 1 + 6 + 11 + 16 + \ldots + (5k - 4) + (5(k+1) - 4) &= \frac{k(5k-3)}{2} + (5(k+1) - 4) \\ &= \frac{k(5k-3) + 2(5(k+1)-4)}{2} \\ &= \frac{5k^2 - 3k + 2(5k+5-4)}{2} \\ &= \frac{5k^2 - 3k + 10k + 10 - 8)}{2} \\ &= \frac{5k^2+7k+2}{2} \\ \text{which as noticed above} \quad &= \frac{(k+1)(5(k+1)-3)}{2} \end{aligned}$$

$\square$

8. (Induction Proof) Let n be a positive integer. For all $n \geqslant 3$ let $P(n)$ be the formula

$$4^3 + 4^4 + 4^5 + \ldots + 4^n = \frac{4(4^n - 16)}{3}$$

(a) *(Base Case)*. Show that $P(3)$ is true. *[4]*

$$\begin{aligned} 4^3 &= \frac{4(4^3 - 16)}{3} \\ 64 &= \frac{4(64 - 16)}{3} \\ &= \frac{4(64 - 16)}{3} \\ &= \frac{256 - 64}{3} \\ &= \frac{192}{3} \\ 64 &= 64 \end{aligned}$$

(b) *(Inductive Hypothesis)*. State $P(k)$. *[1]*

$$4^3 + 4^4 + 4^5 + \ldots + 4^k = \frac{4(4^k - 16)}{3}$$

(c) *(Conclusion)*. State $P(k+1)$. *[1]*

$$4^3 + 4^4 + 4^5 + \ldots + 4^n + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}$$

(d) *(Inductive Step)*. Show that $P(k) \rightarrow P(k+1)$. *[10]*
By the induction hypothesis, let

$$4^3 + 4^4 + 4^5 + \ldots + 4^k = \frac{4(4^k - 16)}{3}$$

Add 4^{k+1} to both sides

$$\begin{aligned} 4^3 + 4^4 + 4^5 + \ldots + 4^k + 4^{k+1} &= \frac{4(4^k - 16)}{3} + 4^{k+1} \\ &= \frac{4(4^k - 16) + 3(4^{k+1})}{3} \\ &= \frac{4 \cdot 4^k - 4 \cdot 16 + 3 \cdot 4^{k+1}}{3} \\ &= \frac{4^{k+1} - 4 \cdot 16 + 3 \cdot 4^{k+1}}{3} \\ &= \frac{4 \cdot 4^{k+1} - 4 \cdot 16}{3} \\ &= \frac{4(4^{k+1} - 16)}{3} \end{aligned}$$

$\square$