CS128 Spr2012 Exam 3 KEY

This is a closed-book, closed-notes exam. Mark each sheet of paper you use with your name and the string "cs128 Spr2012 exam#3". If you are caught cheating, you will receive a zero grade for this exam. The number of points each question is worth is indicated in square brackets after each question. The sum of the points for all the questions is 55, but note that the exam's score will be capped at 50 (i.e., there are 5 bonus points but you can't score more than 50). You have exactly 50 minutes to complete this exam. Keep your answers <u>clear and concise</u> while complete. Simplify all formulas down to powers and the four basic operations. Reminder: $\mathbb{Z}$ is the set of all integers. Good luck!

1. Let f be a function from $\mathbb{Z}$ to $\mathbb{Z}$ defined as $f(n) = 6n - 8$.

   (a) Is f one-to-one? If not, give a counterexample. *[2]*
   Yes

   (b) Is f onto? If not, give a counterexample. *[2]*
   No
   Although the function is linear, it is a function over *Integers*.
   Counterexample: There is no integer n for which $f(n)$ equals 0, 1, or 2.

2. Let R be a relationship on $\mathbb{Z}$ defined as $R = \{(a, b) \mid 3 \text{ divides } (a - b)\}$.

   (a) Is R Reflexive? If not, give a counterexample. *[2]*
   Yes
   Note that 3 divides 0.

   (b) Is R Symmetric? If not, give a counterexample. *[2]*
   Yes

   (c) Is R Transitive? If not, give a counterexample. *[2]*
   Yes

3. Consider the string $s =$ "JAMESTOWN".

   (a) In how many different ways can the letters of s be rearranged?. *[3]*

   $$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

   (b) How many such arrangements contain the substring "TOWN". *[3]*

   If we consider the string "TOWN" as a single element, then what we need is to count the number of permutations of a set of 6 elements.

   $$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

4. Usernames in a famous university consists of 7 characters: 3 letters (based on the user's name) followed by 4 *non-repeating* letters or digits (with no significant meaning).

   (a) How many different usernames are possible? *[3]*

   Given that there are 26 letters and 10 digits. Be first three characters of a username can be built in 26^3 ways (as repetitions are allowed). The last 4 characters can be built in $36 \cdot 35 \cdot 34 \cdot 33$ ways. Therefore the answer is:

$$26^3 \cdot 36 \cdot 35 \cdot 34 \cdot 33 = 24,847,542,720$$

   (b) Graduates students are assigned usernames where the fourth character is the digit '0'. How many graduate student usernames are possible? *[3]*

   As there is only 1 choice for the 4th character:

$$26^3 \cdot 1 \cdot 35 \cdot 34 \cdot 33 = 690,209,520$$

   (c) The IT department has a list of 89 reserved 4 letter words. Accounts whose last 4 characters is a reserved word are deemed invalid. How many valid usernames are possible? *[3]*

   If we see this problem as a 2 step operation. Step 1 is to build the first 3 characters. Step 2 is to build the last 4 non-repeating characters. We now need to remove 86 options from step 2.

$$26^3 \cdot [(36 \cdot 35 \cdot 34 \cdot 33) - 86] = 24,846,031,184$$

5. How many different solutions does the equation $x_1 + x_2 + x_3 + x_4 = 115$ has if every x_i has a value in the range [5..100] inclusive. *[5]*

   We can see the left side of the equation as a unary string of $115 + 3$ characters. However, as each x_i is at least 5 we are forced to have 5 charactes of each x_i filled with '1's.
   Hence, we only have $115 - (5 * 4) + 3 = 98$ places where to put the rest of the '1's and the 3 +'s. So the solution is:

$$\binom{98}{3} \text{ or } \binom{98}{95}$$

$$\binom{98}{3} = \frac{98!}{3!(98-3)!} = \frac{98 \cdot 97 \cdot 96 \cdot 95!}{3! \cdot 95!} = \frac{98 \cdot 97 \cdot 96}{6} = 152,096$$

6. 5 people have been exposed to the Zombie virus. A person has a 50-50 chance to become a Zombie once exposed.

   (a) What is the probability that none of the 5 people turns into a zombie? *[3]*

   Notice that this is similar to tossing 5 coins. There are 2^5 possible outcomes, so the size of the sample space S is 2^5.
   Let E ve the event that no one turns into a Zombie, $E = \{(HHHHH)\}$. So:

$$p(E) = \frac{|E|}{|S|} = \frac{1}{2^5}$$

(b) What is the probability that exactly 3 people turn into zombies? *[3]*

Let E be the event that exactly 3 people turn into zombies. The number of outcomes in E can be found by considering how many ways we can choose 3 people out of 5 to turn into zombies. Therefore:

$$|E| = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!} = 5 \cdot 2 = 10$$

So:

$$p(E) = \frac{|E|}{|S|} = \frac{10}{2^5}$$

(c) What is the probability that at least 4 people turn into zombies? *[3]*

Let E be the event that at least 4 people turn into zombies. E can be partitioned into two disjoint parts:

$$E = \text{Exactly 4 Zombies } \cup \text{Exactly 5 Zombies}$$

Therefore, using the addition rule:

$$|E| = \binom{5}{4} + \binom{5}{5} = \frac{5!}{4!(5-4)!} + 1 = \frac{5 \cdot 4!}{4! \cdot 1!} + 1 = 5 + 1 = 6$$

So:

$$p(E) = \frac{|E|}{|S|} = \frac{6}{2^5}$$

7. A raffle sells 1 million tickets at \$5 each. The raffle has a first prize of \$1,000,000, 3 second prizes of \$100,000 each and 6 third prizes of \$10,000 each. What is a contestant's expected gain or loss? (What is the expected value of the random "variable" f defined as the prize of a ticked minus the cost of a ticket?) *[8]*

   Recall that the expected value of a random variable f is equal to the sum for every outcome c of its probability times its value. Applying this formula directly to the problem yields:

$$\text{Exp}(f) = \frac{1}{10^6} \cdot (10^6 - 5) + \frac{3}{10^6} \cdot (10^5 - 5) + \frac{6}{10^6} \cdot (10^4 - 5) + \frac{10^6 - 10}{10^6} \cdot (-5)$$

8. A fridge contains 25 sodas: 5 diet colas, 8 regular colas, 3 diet lemon-limes and 9 regular lemon-limes. Bob picks up a soda at random.

   (a) If Bob picks a diet soda, what is the probability he picked a cola? *[4]*

$$p(\text{Cola}|\text{Diet}) = \frac{p(\text{Cola} \cap \text{Diet})}{p(\text{Diet})} = \frac{\frac{5}{25}}{\frac{8}{25}} = \frac{5 \cdot 25}{8 \cdot 25} = \frac{5}{8}$$

   (b) If Bob picks a lemon-lime soda, what is the probability he picked a diet soda? *[4]*

$$p(\text{Diet}|\text{Lime}) = \frac{p(\text{Diet} \cap \text{Lime})}{p(\text{Lime})} = \frac{\frac{3}{25}}{\frac{12}{25}} = \frac{3 \cdot 25}{12 \cdot 25} = \frac{3}{12}$$