INTRODUCTION TO TOPOPOGY

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Chapter 1: Topological Spaces

Suppose that X is a set. Then the statement that T is a topology for X means that T is a collection of subsets of X such that:

(i) $X \in T$ and $\emptyset \in T$,

(ii) The union of the sets of any subcollection of T is an element of T (i.e. if $S \subset T$, then $[\cup\{V : V \in S\}] \in T$), and

(iii) If $U \in T$ and $V \in T$, then $U \cap V \in T$.

The statement that Σ is a topological space means that Σ is an ordered pair (X, T), where X is a set and T is a topology for X. If (X, T) is a topological space Σ, then U is an open set in Σ if and only if $U \in T$; p is a point of Σ if and only if $p \in X$; and A is a subset of Σ if and only if $A \subset X$.

The following examples demonstrate the abundance of topological spaces. The proof that each is a topological space is left as an exercise.

Example 1.1: Let $\mathbb{R}^1$ be the set of all real numbers. An open interval in $\mathbb{R}^1$ is a set of the form $\{x : a < x < b\}$. Let T be the collection of all subsets of $\mathbb{R}^1$ having the property that if $U \in T$ and $x \in U$ then there is an open interval I such that $x \in I$ and $I \subseteq U$. Then T is a topology for $\mathbb{R}^1$. T is called the usual topology for $\mathbb{R}^1$, and the topological space $(\mathbb{R}^1, T)$ is denoted E^1.

Example 1.2: Let X be a set, and let T be the collection of all subsets of X. Then T is a topology for X called the discrete topology, and the topological space (X, T) is called a discrete topological space.

Example 1.3: Let X be a set and T be $\{X, \emptyset\}$. Then T is a topology for X called the indiscrete topology, and the topological space (X, T) is called an indiscrete topological space.

Example 1.4: Let X be a set. A topology T for X is defined as follows: Suppose $U \subset X$; then $U \in T$ if and only if either (i) $U = \emptyset$, (ii) $U = X$, or (iii) there exists a finite set F in X such that $U = X - F$ (i.e. $U = \{x \in X : x \notin F\}$). T is called the finite complement topology for X.

Example 1.5: Let $\mathbb{R}^n$ be the set of all n-tuples of real numbers. Recall that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then the distance, $d(x, y)$, between x and y is usually defined as $d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$. A topology T for $\mathbb{R}^n$ can be defined as follows: a subset U of $\mathbb{R}^n$ belongs to T if and only if for each point p of U, there is a positive number ϵ such that $\{x : d(x, p) < \epsilon\}$ is a subset of U. T is called the usual topology for $\mathbb{R}^n$, and the topological space $(\mathbb{R}^n, T)$ is denoted E^n.

The reader may show that we have not abused notation in Example 1.5. That is, the topological space E^1 defined in Example 1.1 and the topological space E^n with $n = 1$ defined in Example 1.5 are the same. To do this, one must show that the set of open sets defined in each example are equal.

The following definitions are necessary for an understanding of the elementary theorems about topological spaces.

Suppose that (X, T) is a topological space, $A \subset X$, and $p \in X$. The statement that p is a limit point of A means that if U is an open set and $p \in U$, then U contains a point of A distinct from p. A is closed if and only if every limit point of A belongs to A. The closure of A, denoted $\overline{A}$, is the union of A and the set of limit points of A. For instance, in the topological space E^1, let $A = [0, 1/2) \cup (1/2, 1]$. Then $1/2$ is a limit point of A, and therefore A is not closed. Further, $\overline{A} = [0, 1]$. Notice that a subset of a topological space may be neither open nor closed as, for example, the subset $[0, 1)$ is neither open nor closed in E^1.

Question: What do the closed sets "look like" in Examples 1.1–1.4?

Theorem 1.1: Let (X, T) be a topological space, and let M be a subset of X. Then M is closed if and only if $X - M$ is open.

Theorem 1.2: If (X, T) is a topological space, the union of finitely many closed subsets of X is closed.

Theorem 1.3: If (X, T) is a topological space, and L is a collection of closed subsets of X, then $\cap\{C : C \in L\}$ is a closed subset of X.

Theorem 1.4: If M is a subset of a topological space, then $\overline{M} = \overline{\overline{M}}$. Hence $\overline{M}$ is closed.

Theorem 1.5: If (X, T) is a topological space, $M \subset X$, $K \subset M$, and p is a limit point of K, then p is a limit point of M.

Theorem 1.6: If (X, T) is a topological space, F is a finite collection of subsets of X, $p \in X$, and p is a limit point of the union of the sets in F, then p is a limit point of some set in F.

Theorem 1.7: Suppose that (X, T) is a topological space, n is a positive integer, and each of $A_1, A_2, \dots, A_n$ is a subset of X. Then

$$\bigcup_{k=1}^{n} \overline{A_k} = \overline{\bigcup_{k=1}^{n} A_k}.$$

The following concept is extremely useful and important throughout the study of topology.

Suppose that X is a set, that T is a topology for X, and that $B \subset T$. Then the statement that B is a basis or base for the topology T means that if $U \in T$ and $p \in U$, then there exists a set $V \in B$ such that $p \in V$ and $V \subset U$.

Theorem 1.8: If (X, T) is a topological space and B is a basis for the topology T, then for any $U \in T$ there is $A \subset B$ such that $U = \cup\{V : V \in A\}$.

Previous examples have shown that the collection of all open interval in $\mathbb{R}^1$ is a basis for the topology E^1, and the collection of all sets of the form $\{x : \mathrm{d}(p, x) < \epsilon\}$ in $\mathbb{R}^n$ is a basis for the topology of E^n. Of course, a given topology will not have a unique basis; indeed, if T is a topology, B is a basis for T, and $B \subset B_1 \subset T$, then B_1 is a basis for T. It is frequently useful to find a "small" basis for a given topology. The reasons for this will be discussed later.

Now consider a slightly different problem. Suppose that X is a set and B is a collection of subset of X, what conditions must we impose on B in order that B will serve as a basis for a topology for X? Theorem 1.9 answers this question. The reader should convince themself that if B is a basis for a topology for X, then that topology is unique; in other words, if B is a basis for topologies T_1 and T_2, then $T_1 = T_2$.

Theorem 1.9: If X is a set and B is a collection of subsets of X, then B is a basis for a topology for X if and only if the following conditions hold:

(a) X is the union of the sets in B, and

(b) if U and V are sets in B, and $p \in U \cap V$, then there is a set W in B such that $p \in W$ and $W \subset U \cap V$.

Theorem 1.9 may be used to obtain simple descriptions of some interesting topological spaces. In each of the following the reader should verify that the conditions of Theorem 1.9 are satisfied.

Example 1.6: Let $\mathbb{R}^1$ be the set of real numbers, and let B be the collection of subsets of $\mathbb{R}^1$ of the form $\{x : a \leq x < b\}$. Then B is a basis for a topology T for $\mathbb{R}^1$. The topological space $(\mathbb{R}^1, T)$ will be called "E^1 bad".

Example 1.7: Let $\mathcal{Q}$ be the set of rational numbers. Suppose that $r \in \mathcal{Q}$, that n is a positive integer, and that ϵ is a positive number less than $\pi/4$. Let $A(r, n, \epsilon)$ be the intersection of $\mathcal{Q}$ with $\{x : (r + n\pi) - \epsilon < x < (r + n\pi) + \epsilon\}$. Let $B(r, \epsilon)$ be $\bigcup_{n=1}^{\infty} A(r, n, \epsilon)$, and let $C(r, \epsilon)$ be $\{r\} \cup B(r, \epsilon)$. Now let B be the collection of all sets of the form $C(r, \epsilon)$. Then B is a basis for a topology for $\mathcal{Q}$. Example 1.7 is $(\mathcal{Q}, \mathcal{T})$.

Example 1.8: Let X be a set with a total order relation $<$, and let B be the collection of all subsets of X of one of the three following forms: $\{x : p < x\}$, $\{x : x < p\}$, or $\{x : a < x < b\}$. Then B is a basis for a topology T for X. T is called the order topology for X.

A relation on a set X is a subset R of $X \times X = \{(x, y) : x \in X \text{ and } y \in X\}$. A total order relation is a relation which satisfies the following properties:

(i) For any $x, y \in X$ with $x \neq y$ either $(x, y) \in R$ or $(y, x) \in R$.

(ii) If $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

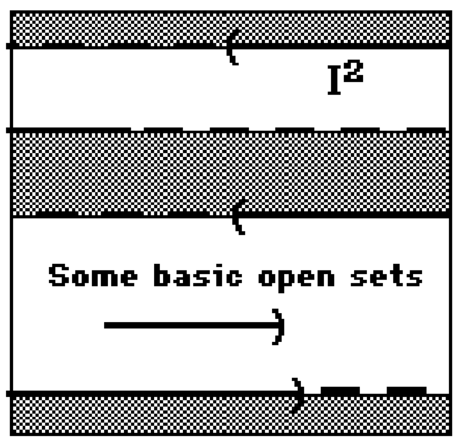
(iii) $(x, x) \notin R$.

(iv) $(x, y) \in R$ if and only if $(y, x) \notin R$.

Property (iv) is actually a consequence of the preceding properties.

Usually, one denotes an order relation by an inequality sign $<$. That is, if R is an order relation on X, we denote $(x,y) \in R$ by $x < y$.

Example 1.9: Let I^2 be the unit square in $\mathbb{R}^2$. A total order relation $<$ is defined on I^2 as follows: $(x,y) < (x',y')$ if and only if either (i) $y < y'$, or (ii) $y = y'$ and $x < x'$. Let T be the order topology. Then (I^2, T) is Example 1.9. This topological space well be called the lexicographically ordered square.



Example 1.10: Let $X = \{a,b,c\}$, and let B be $\{\{a,c\},\{b,c\},\{c\}\}$. Then B is a basis for a topology T for X. Example 1.10 is (X,T).

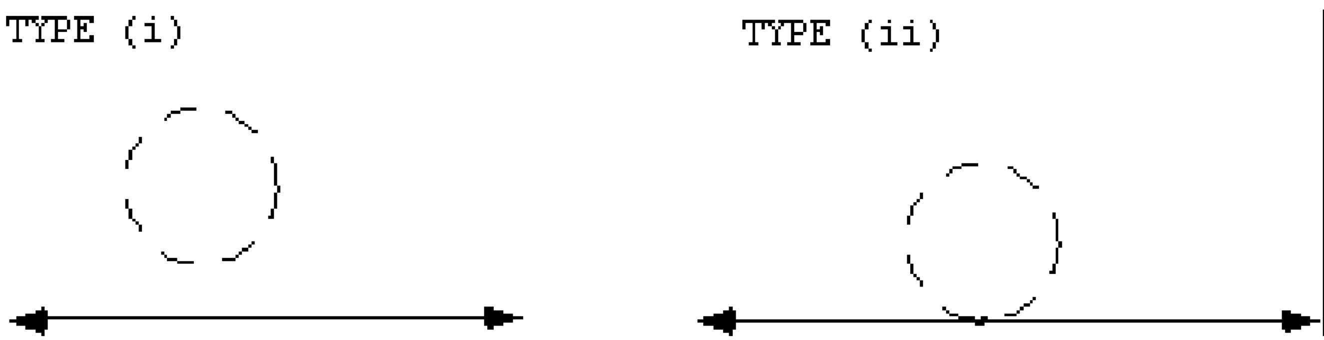
A set X is said to be well ordered by a total order relation $<$ if and only if A is any nonempty subset of X then there is an element p in A such that for all $x \in A$, $p < x$ or $p = x$.

Example 1.11: Let X be the set of all ordinal numbers less than or equal to the first uncountable ordinal; or, equivalently, suppose that X is an uncountable well ordered set with a largest element Ω such that if $\alpha < \Omega$, then $\{x : x < \alpha\}$ is countable. Let T be the order topology for X. Example 1.11 will be denoted $[1, \Omega]$, and will be called the closed ordinal space.

Example 1.12: Let $\mathbb{R}^2_+$ be $\{(x,y): (x,y) \in \mathbb{R}^2 \text{ and } y \geq 0\}$. Let B be the collection of all subsets of $\mathbb{R}^2_+$ of the following types:

(i) interiors of circles which are disjoint from the x-axis; and

(ii) interiors of circles tangent to the x-axis from above, together with the point of tangency.



Then B is a basis for a topology T for $\mathbb{R}^2_+$. the topological space $(\mathbb{R}^2_+, T)$ will be called the upper half plane space.

Example 1.13: In $\mathbb{R}^2$, let B be the collection of all subsets which are obtained by removing a countable set from the interior of a circle in $\mathbb{R}^2$. Then B is a basis for a topology T for $\mathbb{R}^2$, and Example 1.13 is $(\mathbb{R}^2, T)$.

A method for using old topological spaces to find new ones is shown in the following theorem.

Theorem 1.10: Suppose that (X,T) is a topological space, $Y \subset X$, and $S = \{U$: for some set V of T, $U = Y \cap V\}$. Then S is a topology for Y.

Example 1.14: Example 1.14 is the subspace of Example 1.11, obtained by deleting the largest element of X. This example will be denoted by $[1, \Omega)$, and will be called the half-open ordinal space.

Let us next devise one more technique for constructing new spaces from old. Let (X_1, T_1) and (X_2, T_2) be topological spaces. To use the topologies T_1 and T_2 to construct a topology for the Cartesian product $X_1 \times X_2 = \{(x, y): x \in X_1 \text{ and } y \in X_2\}$, let $B = \{U \times V: U \in T_1 \text{ and } V \in T_2\}$. It is left to the reader to verify that B is a basis of a topology T for $X_1 \times X_2$. T is called the product topology, and the topological space $(X_1 \times X_2, T)$ is called the product of (X_1, T_1) and (X_2, T_2). A later chapter will discuss techniques for generalizing this process to form the product of infinitely many topological spaces.

Example 1.15: Example 1.15 is "E^1 bad" $\times$ "E^1 bad".

Question How are "E^1 bad" $\times$ "E^1 bad" and E^n with $n = 2$ related?