Int. J. Mech. Sci. Vol. 35, No. 5, pp. 387-396, 1993
Printed in Great Britian.

0020-7403/93 $6.00 + .00

ACTIVE CONTROL OF COMPOSITE PLATES USING PIEZOELECTRIC STIFFENERS

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(*Received* 8 *June* 1992; *and in revised form* 6 *November* 1992)

Abstract—The governing equations for geometrically nonlinear, arbitrarily laminated rectangular plates reinforced by stiffeners which include piezoelectric and composite layers are presented. General equations obtained in the paper are reduced to a single equation of motion for piezoelectrically reinforced, geometrically linear, specially orthotropic plates. A criterion for an effective control of forced vibrations of such plates using piezoelectric stiffeners and a static electric field is illustrated. Active control of dynamic stability using a dynamic electric field with a frequency equal to that of the in-plane load is also considered.

In addition, an approach to the analysis of piezoelectrically stiffened nonlinear plates whose motion is represented by single-term functions of the coordinates is discussed.

Numerous active control problems can be addressed using the theory outlined in the paper.

NOTATION

A_r, A_s	cross-sectional areas of the stiffeners oriented in the y- and x-directions, respectively
A_{ij}	extensional stiffnesses of the plate
A_{sc}, A_{rc}	cross-sectional areas of composite layers in the stiffeners
A_{sp}, A_{rp}	cross-sectional areas of piezoelectric layers in the stiffeners
B_{ij}	coupling stiffnesses of the plate
D_{ij}	bending stiffnesses of the plate
d_{31}	piezoelectric constant
E_r, E_s	moduli of elasticity of composite layers in the axial directions of the stiffeners oriented in the y- and x-directions, respectively
E_p	modulus of elasticity of piezoelectric layers
F_{sc}, F_{rc}	first moments of cross-sectional areas of composite layers in the stiffeners about the plate middle plane
F_{sp}, F_{rp}	first moments of cross-sectional areas of piezoelectric layers in the stiffeners about the plate middle plane
h	plate thickness
I_{sc}, I_{rc}	second moments of cross-sectional areas of composite layers in the stiffeners about the plate middle plane
I_{sp}, I_{rp}	second moments of cross-sectional areas of piezoelectric layers in the stiffeners about the plate middle plane
N_i	in-plane stress resultants ($i = 1, 2, 6$)
M_i	stress couples ($i = 1, 2, 6$)
t_x, t_y	thicknesses of piezoelectric layers in the stiffeners oriented in the x- and y-directions, respectively
u, v, w	in-plane and transverse displacements
V	voltage
V_0	static voltage
x_r	coordinates of the stiffeners oriented in the y-direction
y_s	coordinates of the stiffeners oriented in the x-direction
z_{sp}, z_{rp}	distances from the cross-section centroids of piezoelectric layers to the plate middle plane
ω	frequency of the driving force

INTRODUCTION

Development of smart composite structures represents one of the most significant recent trends in the mechanics of structures. These structures can provide significant advantages as compared with traditional "passive" structures. Possible active components which can be used as a part of an overall composite material can include magnetostrictive and electrostrictive materials, shape memory alloys, etc. Another promising opportunity is presented by piezoelectric materials.

The first investigations of piezoelectric structures were published by Haskins and Walsh [1] and Toupin [2] who considered cylindrical and spherical shells, respectively. Linear vibrations of piezoelectric plates were analyzed in a monograph by Tiersten [3]. Adelman and Stavsky [4] published a pioneering paper where they investigated vibrations of composite cylinders and disks formed by piezoceramic and metallic layers. Developments in the area of application of piezoelectric materials in smart structures were outlined by Rogers *et al.* [5, 6] and Hanagud *et al.* [7].

Recent studies dealing with identification and control of anisotropic plates and shells using piezoelectric sensors and actuators have been published by Hagood *et al.* [8], Crawley and Lazarus [9], Tzou and Garde [10], Tzou and Zhong [11] and Sung *et al.* [12].

The references listed above considered distributed sensors and actuators which can be either embedded in the structure or bonded to its surface. It may be beneficial to arrange the actuators (and, possibly, the sensors) in the traditional pattern of stiffeners. In the present paper, governing equations are developed for a geometrically nonlinear, generally laminated rectangular plate reinforced by two mutually perpendicular systems of stiffeners, each system being parallel to the plate edges (Fig. 1). The stiffeners can include a piezoelectric layer and composite material layers as shown in Fig. 1.

Analysis: governing equations

Governing equations for geometrically nonlinear, generally laminated composite plates reinforced by the stiffeners composed of the layers of composite and piezoelectric materials are formulated in this section. The stiffeners are assumed to be perfectly bonded to the plate. This means that deformations of the plate and the stiffeners are continuous.

Consider a rectangular plate reinforced by stiffeners oriented in the x- and y-directions (Fig. 1). Equations of motion obtained by the assumption that in-plane inertial terms are negligible read:

$$N_{1,x} + N_{6,y} = 0,$$
$$N_{6,x} + N_{2,y} = 0,$$
$$M_{1,xx} + 2M_{6,xy} + M_{2,yy} + (N_1 w_{,x} + N_6 w_{,y})_{,x} + (N_6 w_{,x} + N_2 w_{,y})_{,y} = \rho w_{,tt}, \tag{1}$$

where:

$$\begin{Bmatrix} N_1 \\ M_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_x \begin{Bmatrix} 1 \\ z \end{Bmatrix} \mathrm{d}z + \sum_s \delta(y - y_s) \int_{A_s} \sigma_x \begin{Bmatrix} 1 \\ z \end{Bmatrix} \mathrm{d}A_s,$$
$$\begin{Bmatrix} N_2 \\ M_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_y \begin{Bmatrix} 1 \\ z \end{Bmatrix} \mathrm{d}z + \sum_r \delta(x - x_r) \int_{A_r} \sigma_y \begin{Bmatrix} 1 \\ z \end{Bmatrix} \mathrm{d}A_r,$$
$$\begin{Bmatrix} N_6 \\ M_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{xy} \begin{Bmatrix} 1 \\ z \end{Bmatrix} \mathrm{d}z. \tag{2}$$

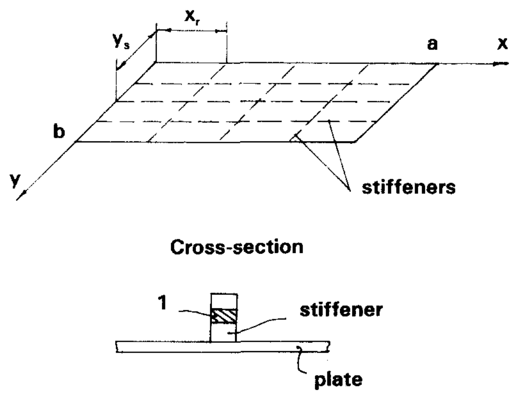

FIG. 1. Geometry of the plate; 1 = piezoelectric layer.

In Eqns (1) and (2) σ_x, σ_y and σ_{xy} are in-plane stresses and the coordinate z is measured from the middle surface of the plate. The effect of the stiffeners is accounted for via Dirac's delta function $\delta(\ldots)$. In a particular case of a large number of closely spaced identical stiffeners in one or both directions, the so-called smeared stiffeners technique can be used. According to this technique:

$$\delta(x - x_r) = \frac{1}{l_r}$$

$$\delta(y - y_s) = \frac{1}{l_s}, \tag{3}$$

l_r and l_s being the spacings of the corresponding stiffeners.
The mass of the plate per unit area is:

$$\rho = \bar{\rho}h + \sum_r \delta(x - x_r)\rho_r A_r + \sum_s \delta(y - y_s)\rho_s A_s, \tag{4}$$

where $\bar{\rho}$, ρ_r and ρ_s are the average mass densities of the materials of the plate and the stiffeners.

Constitutive relationships given by Eqn (2) can be subdivided into the contributions of the plate and those of the stiffeners. The plate relationships for a generally laminated plate with geometrically nonlinear deformations are described by von Karman type plate theory:

$$\begin{Bmatrix} N'_1 \\ N'_2 \\ N'_6 \\ M'_1 \\ M'_2 \\ M'_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & & & & D_{22} & D_{26} \\ sym & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} u_{,x} + \frac{1}{2}w_{,x}^2 \\ v_{,y} + \frac{1}{2}w_{,y}^2 \\ u_{,y} + v_{,x} + w_{,x}w_{,y} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \tag{5}$$

where a prime denotes the plate contributions.

The contribution of the stiffeners is obtained using physically linear electromechanical constitutive equations [3]. In the present problem, the plate is subject to a voltage V in the z-direction resulting in axial strains in the piezoelectric layers of the stiffeners. Accordingly, the axial stresses in these layers in the stiffeners oriented along the x- and y-axes are [13]:

$$\sigma_x = E_p\left(u_{,x} + \frac{1}{2}w_{,x}^2 - zw_{,xx} - \frac{d_{31}V}{t_x}\right) \tag{6}$$

$$\sigma_y = E_p\left(v_{,y} + \frac{1}{2}w_{,y}^2 - zw_{,yy} - \frac{d_{31}V}{t_y}\right). \tag{7}$$

Obviously, the stresses in the layers of composite materials that form the stiffeners can be obtained from Eqns (6) and (7) by deleting the voltage-dependent terms and using the appropriate modulus of elasticity.

Now the stress resultants and couples contributed by the stiffeners can be obtained:

$$N''_1 = \sum_s \delta(y - y_s)\left[(E_pA_{sp} + E_sA_{sc})\left(u_{,x} + \frac{1}{2}w_{,x}^2\right) - (E_pF_{sp} + E_sF_{sc})w_{,xx} - E_pA_{sp}\frac{d_{31}}{t_x}V\right],$$

$$M''_1 = \sum_s \delta(y - y_s)\left[(E_pF_{sp} + E_sF_{sc})\left(u_{,x} + \frac{1}{2}w_{,x}^2\right) - (E_pI_{sp} + E_sI_{sc})w_{,xx} - E_pF_{sp}\frac{d_{31}}{t_x}V\right],$$

$$N_2'' = \sum_r \delta(x - x_r)\left[(E_p A_{rp} + E_r A_{rc})\left(v_{,y} + \frac{1}{2} w_{,y}^2\right) - (E_r F_{rp} + E_r F_{rc}) w_{,yy} - E_p A_{rp} \frac{d_{31}}{t_y} V\right],$$

$$M_2'' = \sum_r \delta(x - x_r)\left[(E_p F_{rp} + E_r F_{rc})\left(v_{,y} + \frac{1}{2} w_{,y}^2\right) - (E_p I_{rp} + E_r I_{rc}) w_{,yy} - E_p F_{rp} \frac{d_{31}}{t_y} V\right]. \tag{8}$$

Total stress resultants and stress couples are obtained by superposition of the contributions of the plate and those of the stiffeners:

$$N_i = N_i' + N_i'', \quad M_i = M_i' + M_i'', \tag{9}$$

where $i = 1, 2, 6$.

The substitution of Eqns (9), (5) and (8) into Eqn (1) yields a system of nonlinear differential equations which can be presented in the operational–functional form as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \begin{bmatrix} 0 & 0 & K_{13}(w) \\ 0 & 0 & K_{23}(w) \\ K_{31}(u, w) & K_{32}(v, w) & K_{33}(w) \end{bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ N_{33}(w) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \rho w_{,tt} \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} V + \begin{Bmatrix} 0 \\ 0 \\ PW(w, V) \end{Bmatrix}. \tag{10}$$

In Eqns (10), $[L_{ij}]$ is a matrix of linear differential operators; the terms representing quadratic and cubic nonlinearities are assembled in the matrix $[K_{ij}]$ and the function $N_{33}(w)$, respectively, $\{P_i\}$ is a vector of linear differential operators, and $PW(w, V)$ is a linear function of w with the coefficients depending on the voltage V.

The linear operators in Eqns (10) are:

$$L_{11} = \bar{A}_{11} \frac{\partial^2}{\partial x^2} + 2A_{16} \frac{\partial^2}{\partial x \partial y} + A_{66} \frac{\partial^2}{\partial y^2},$$

$$L_{12} = L_{21} = A_{16} \frac{\partial^2}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2}{\partial x \partial y} + A_{26} \frac{\partial^2}{\partial y^2},$$

$$L_{13} = -L_{31} = -\left[\bar{B}_{11} \frac{\partial^3}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x \partial y^2} + 3B_{16} \frac{\partial^3}{\partial x^2 \partial y} + B_{26} \frac{\partial^3}{\partial y^3}\right],$$

$$L_{22} = A_{66} \frac{\partial^2}{\partial x^2} + 2A_{26} \frac{\partial^2}{\partial x \partial y} + \bar{A}_{22} \frac{\partial^2}{\partial y^2},$$

$$L_{23} = -L_{32} = -\left[B_{16} \frac{\partial^3}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^2}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3}{\partial x \partial y^2} + \bar{B}_{22} \frac{\partial^3}{\partial y^3}\right],$$

$$L_{33} = -\left[\bar{D}_{11} \frac{\partial^4}{\partial x^4} + 4D_{16} \frac{\partial^4}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4}{\partial x \partial y^3} + \bar{D}_{22} \frac{\partial^4}{\partial y^4}\right], \tag{11}$$

$$P_1 = \sum_s \delta(y - y_s) R_1 \frac{\partial}{\partial x},$$

$$P_2 = \sum_r \delta(x - x_r) R_2 \frac{\partial}{\partial y},$$

$$P_3 = \sum_s \delta(y - y_s) R_1 z_{sp} \frac{\partial^2}{\partial x^2} + \sum_r \delta(x - x_r) R_2 z_{rp} \frac{\partial^2}{\partial y^2}, \tag{12}$$

$$PW(w, V) = \sum_s \delta(y - y_s) R_1 (Vw_{,x})_{,x} + \sum_r \delta(x - x_r) R_2 (Vw_{,y})_{,y}, \tag{13}$$

where the terms with an overbar indicate extensional, coupling and bending stiffnesses affected by the presence of stiffeners:

$$\bar{A}_{11} = A_{11} + \sum_s \delta(y - y_s)(E_p A_{sp} + E_s A_{sc}),$$

$$\bar{A}_{22} = A_{22} + \sum_r \delta(x - x_r)(E_p A_{rp} + E_r A_{rc}),$$

$$\bar{B}_{11} = B_{11} + \sum_s \delta(y - y_s)(E_p F_{sp} + E_s F_{sc}),$$

$$\bar{B}_{22} = B_{22} + \sum_r \delta(x - x_r)(E_p F_{rp} + E_r F_{rc}),$$

$$\bar{D}_{11} = D_{11} + \sum_s \delta(y - y_s)(E_p I_{sp} + E_s I_{sc}),$$

$$\bar{D}_{22} = D_{22} + \sum_r \delta(x - x_r)(E_p I_{rp} + E_r I_{rc}). \tag{14}$$

Constants introduced in Eqns (12) and (13) are:

$$R_1 = E_p A_{sp} \frac{d_{31}}{t_x},$$

$$R_2 = E_p A_{rp} \frac{d_{31}}{t_y}. \tag{15}$$

Nonlinear functions in Eqns (10) are given in the Appendix.

Applications of the developed theory to various active control problems are illustrated in the following examples.

Example 1. free and forced vibrations of specially orthotropic stiffened plates

If linear vibrations of specially orthotropic stiffened plates are analyzed, governing Eqns (10)–(14) are simplified, i.e.:

(a) nonlinear terms in Eqns (10) are omitted; and
(b) stiffnesses $A_{16} = A_{26} = D_{16} = D_{26} = B_{ij} = 0$ in Eqns (11) and (14).

Note that the stiffnesses $\bar{B}_{11}$ and $\bar{B}_{22}$ are equal to zero only if the first moments of the piezoelectric and composite layers of the stiffeners are equal to zero. This is possible, if identical stiffeners are attached to opposite surfaces of the plate. An example of a similar arrangement can be found in Refs [9] and [13], where piezoelectric elements are bonded to opposite surfaces of an elastic substrate. If this is the case, and the stiffeners on the opposite surfaces of the plate are subjected to in-phase voltages, then Eqns (10) yield a single equation for transverse vibrations:

$$\rho \frac{\partial^2 w}{\partial t^2} + \bar{D}_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \bar{D}_{22} \frac{\partial^4 w}{\partial y^4} + P_3 V + PW(w, V) = 0. \tag{16}$$

Obviously, if the voltage is uniform, the term $P_3 V$ does not affect the natural frequencies and can be omitted. Even in the case of a nonuniform voltage and identical stiffeners attached to the opposite surfaces of the plate, $P_3 V = 0$, since $z_{sp} = z_{rp} = 0$ in Eqns (12). In the absence of the stiffeners and the electric field, Eqn (16) is reduced to the well-known equation for free vibrations of specially orthotropic plates.

If all edges of the plate are clamped, the solution can be obtained in the form:

$$w = W_{mn}(t)\left(1 - \cos\frac{2m\pi x}{a}\right)\left(1 - \cos\frac{2n\pi y}{b}\right), \tag{17}$$

where a and b are the lengths of the plate in the x- and y-directions, respectively, and m and n are integers.

Two electric fields are considered here:

(a) uniform voltage:

$$V = V(t);\ \text{and} \tag{18}$$

(b) electric field corresponding to the mode shape of motion:

$$V = V_{mn}(t)\left(1 - \cos\frac{2m\pi x}{a}\right)\left(1 - \cos\frac{2n\pi y}{b}\right). \tag{19}$$

The substitution of Eqns (17)–(19) into Eqn (16) and the Galerkin procedure yield the following equation of motion for a plate with closely spaced identical stiffeners in both directions (smeared stiffeners technique):

$$9\rho\frac{d^2 W_{mn}}{dt^2} + 16\left[3\bar{D}_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + 3\bar{D}_{22}\left(\frac{n\pi}{b}\right)^4\right]W_{mn}$$
$$- kE_p d_{31}\left[\frac{A_{sp}}{t_x l_s}\left(\frac{m\pi}{a}\right)^2 + \frac{A_{rp}}{t_y l_r}\left(\frac{n\pi}{b}\right)^2\right]W_{mn} V(t) = 0, \tag{20}$$

where $k = 12$ for a uniform electric field and $k = 20$ for an electric field given by Eqn (19). Equation (20) can be used for various control problems. Obviously, in the case of a forced motion it must be complemented by terms corresponding to the driving forces.

We denote the natural frequency of a stiffened plate without piezoelectric materials by ω_{mn} and the frequency of the same plate with piezoelectric layers added to the stiffeners and subjected to a static voltage V_0 by $\omega_{mn}(V_0)$. As follows from Eqn (20), if $V_0 > 0$:

$$\omega_{mn} > \omega_{mn}(V_0). \tag{21}$$

The reversal of the polarization yields the opposite result.

Forced vibrations of a plate with arbitrary boundary conditions are described by the equation:

$$\frac{d^2 W_{mn}}{dt^2} + \omega_{mn}^2(V_0) W_{mn} = q_{mn}\sin\omega t, \tag{22}$$

where q_{mn} is a function of the amplitude of the corresponding harmonic in the double Fourier series which represents the transverse load.

To establish a criterion for an effective active control using a static voltage, compare $W_{mn} = W_{mn}(V_0)$ to $W_{mn} = W_{mn}(V = 0)$.

Obviously, an effective control of vibrations can be achieved if the ratio:

$$p = \frac{W_{mn}(V_0)}{W_{mn}(V_0 = 0)} = \frac{\omega_{mn}^2 - \omega^2}{\omega_{mn}^2(V_0) - \omega^2}, \tag{23}$$

is less than unity.

If inequality (21) is satisfied, the latter requirement means that:

$$\omega_{mn}(V_0) < \omega. \tag{24}$$

Inequality (24) indicates the condition for an effective control of forced vibrations of specially orthotropic plates using piezoelectric stiffeners and a static electric field $V_0 > 0$.

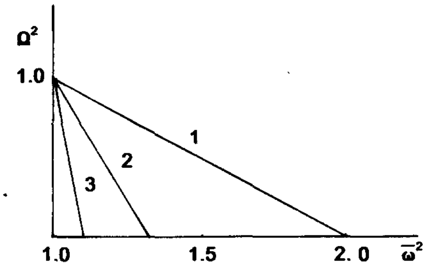

FIG. 2. Conditions for a prescribed level of reduction of the amplitude of forced vibrations. Cases 1, 2 and 3 correspond to $p = 0.50$, 0.25 and 0.10, respectively.

Introducing nondimensional frequencies:

$$\bar{\omega} = \omega/\omega_{mn} \quad \text{and} \quad \Omega = \omega_{mn}(V_0)/\omega_{mn}, \tag{25}$$

one can obtain the amplitude reduction factor in the form:

$$p = \frac{1 - \bar{\omega}^2}{\Omega^2 - \bar{\omega}^2}. \tag{26}$$

The squared ratios Ω necessary to achieve prescribed amplitude reductions are shown in Fig. 2 as functions of squared nondimensional excitation frequencies $\bar{\omega}$. Note that out-of-phase voltages applied to the stiffeners on the opposite surfaces of the plate result in bending moments. Therefore, they can be even more effective for the reduction of forced vibrations.

Example 2. Active control of dynamic instability

Dynamic instability of composite plates was considered by Birman [14], Srinivasan and Chellapandi [15], Bert and Birman [16], Chandiramani and Librescu [17], Tylikowski [18], Moorthy *et al.* [19], Librescu and Thangjitham [20] and Cederbaum [21]. In this example it is shown that in-plane dynamic forces generated by piezoelectric stiffeners can "shift" the regions of instability to "safe" excitation frequencies preventing parametric resonances.

The linear equation of motion of a plate subjected to in-plane loads $N_x^0 \cos 2\omega t$ can be reduced to:

$$\frac{d^2 W_{mn}}{d\tau^2} + \frac{1}{\bar{\omega}^2}[1 + \bar{N}_x \cos 2\tau + \bar{V}(\tau)]\, W_{mn} = 0, \tag{27}$$

where

$$\tau = \omega t, \tag{28}$$

is a nondimensional time.

The expressions for dimensionless in-plane load amplitude, $\bar{N}_x$, and voltage $\bar{V}(\tau)$ which can be obtained for particular boundary conditions and mode shapes of motion are omitted for brevity.

If the frequency of the dynamic voltage is equal to that of excitation, i.e.:

$$\bar{V}(\tau) = \bar{V}\cos 2\tau, \tag{29}$$

Eqn (27) is a Mathieu equation.

According to the theory of Mathieu equations [22], the boundaries of the principal region of dynamic instability can be approximated by the series:

$$\frac{1}{\bar{\omega}^2} = 1 \mp \frac{1}{2\bar{\omega}^2}(\bar{N}_x + \bar{V}) - \frac{1}{32\bar{\omega}^4}(\bar{N}_x + \bar{V})^2 \pm \cdots \tag{30}$$

In particular, if the higher order terms in the right-hand side of Eqn (30) can be neglected:

$$\bar{\omega}^2 = 1 \pm \tfrac{1}{2}(\bar{N}_x + \bar{V}). \tag{31}$$

The analysis of Eqns (30) and (31) illustrates that the parametric resonance (dynamic instability) can be avoided by using piezoelectric effects. This is shown in Fig. 3, which is

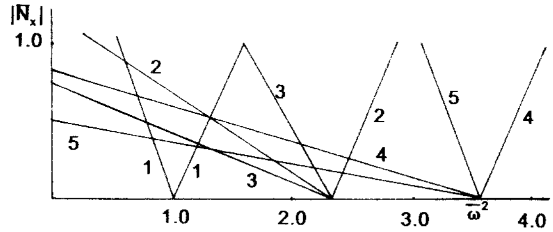

FIG. 3. Principal regions of dynamic instability. Case 1: $\bar{V}=0$ (no voltage); case 2: $\bar{V}=2.5$, $\bar{N}_x\,\bar{V}>0$; case 3: $\bar{V}=2.5$, $\bar{N}_x\,\bar{V}<0$; case 4: $\bar{V}=5.0$, $\bar{N}_x\,\bar{V}>0$; and case 5: $\bar{V}=5.0$, $N_x\,\bar{V}<0$.

obtained using Eqn (31). In this figure, the boundaries of the principal instability region are shown in the axes "squared nondimensional excitation/voltage frequency–nondimensional amplitude of in-plane load".

As follows from Eqns (30) and (31) and Fig. 3, piezoelectric effects result in "shifting" the instability region along the axis of nondimensional excitation frequencies. At the same time, the width of the instability region increases, particularly if $\bar{N}_x$ and $\bar{V}$ have the same sign. It is interesting to note that for the values $\bar{V}<2$ (the case which is not shown in Fig. 3), there are two principal instability regions due to piezoelectric effects. The origin of one of them corresponds to $\bar{\omega}>1$ while the second originates at $\bar{\omega}<1$. This conclusion follows from Eqn (31) where $\bar{N}_x=0$.

Example 3. *Approach to the analysis of active control of nonlinear vibrations*

Consider nonlinear vibrations by the assumption that displacements are single-term functions of coordinates, i.e.:

$$u=\bar{U}(t)f_1(x,y),$$
$$v=\bar{V}(t)f_2(x,y),$$
$$w=\bar{W}(t)f_3(x,y), \tag{32}$$

where $f_i(x,y)$ satisfy the boundary conditions.

An electric field is represented by:

$$V=F(t)f_4(x,y). \tag{33}$$

The substitution of Eqns (32) and (33) into the first two of Eqns (10), and the Galerkin procedure yield:

$$\bar{U}=a_1\,\bar{W}+a_2\,\bar{W}^2+d_1\,F,$$
$$\bar{V}=a_3\,\bar{W}+a_4\,\bar{W}^2+d_2\,F, \tag{34}$$

where $a_i(i=1,2,3,4)$ and $d_j(j=1,2)$ are coefficients.

Upon the substitution of Eqns (32), (33) and (34) and the application of the Galerkin procedure, the last of Eqns (10) becomes:

$$\frac{\mathrm{d}^2\bar{W}}{\mathrm{d}t^2}+(c_1+c_2\,F)\,\bar{W}+c_3\,\bar{W}^2+c_4\,\bar{W}^3+d_3\,F=0, \tag{35}$$

where c_i and d_3 are coefficients.

Equation (35) can be used for the active control analysis of free nonlinear vibrations. Driving forces can be incorporated into this equation.

Notably, if the electric field is static, i.e. $F=$ constant, Eqn (35) is similar to the equation of motion of reinforced composite cylindrical shells suddenly subjected to a thermal field [Eqn (12) in Ref. [23]]. Therefore, the discussion presented in the above paper can be extended to the present case. A periodic motion, which occurs if $c_1+c_2\,F>0$, can be described by:

$$t-t_0=\int_{w_0}^{w^*}\Big[(c_1+c_2\,F)(1-W^{*2})+\tfrac{2}{3}c_3\,\bar{W}_{\max}(1-W^{*3})$$
$$+\tfrac{1}{2}c_4\,\bar{W}^2_{\max}(1-W^{*4})+2(d_3/\bar{W}_{\max})F(1-W^*)\Big]^{-\frac{1}{2}}\,\mathrm{d}W^*, \tag{36}$$

where:

$$W^* = \bar{W}/\bar{W}_{\max}, \quad W_0 = W^*(t_0), \tag{37}$$

t_0 being a reference time.

The right-hand side of Eqn (36) can be represented as an elliptic integral, as shown in application to free vibrations of beams where $c_3 = 0$ by Woinowsky-Krieger [24], Burgreen [25] and others. However, the two-term Galerkin approximation can provide a simple yet accurate solution [26]. Details of such solutions for an equation similar to Eqn (28) can be found in Ref. [23].

CONCLUSIONS

This paper presents a mathematical formulation of the active control problem for arbitrarily laminated rectangular composite plates reinforced by stiffeners which include piezoelectric layers. Geometrically nonlinear equations of motion are presented. In the example considered in the paper, these equations are reduced to a single equation of motion for linear vibrations of specially orthotropic stiffened plates subjected either to a uniform electric field or to a field that reflects the mode shape of vibrations. A criterion for an active control of forced vibrations using piezoelectric stiffeners is illustrated. Control of dynamic instability using dynamic piezoelectric effects is also discussed. In the other example, a nonlinear resonse of a plate is discussed by the assumption that displacements can be represented by single-term functions of the in-plane coordinates.

Acknowledgement—Discussions with Professor Horn-Sen Tzou of the University of Kentucky are warmly appreciated. This research was supported in part by the Army Research Office grant DAAL 03-92-G0301. The author is pleased to acknowledge the assistance of Dr Gary L. Anderson, Member of the Engineering and Environmental Sciences Division of the Army Research Office.

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APPENDIX

Nonlinear functions in Eqns (10)

(1) *Quadratic nonlinearity.*

(a) Operators defined by:

$$K_{i3}(w) = \tilde{K}_{i3}\, w \quad (i = 1, 2),$$

$$\tilde{K}_{i3} = \bar{A}_{11}\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{\partial}{\partial x}\frac{\partial^2}{\partial x\,\partial y} + (A_{12} + A_{66})\frac{\partial}{\partial y}\frac{\partial^2}{\partial x\,\partial y} + A_{16}\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2} + A_{66}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2} + A_{26}\frac{\partial}{\partial y}\frac{\partial^2}{\partial y^2}$$

$$\tilde{K}_{23} = A_{16}\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66})\frac{\partial}{\partial x}\frac{\partial^2}{\partial x\,\partial y} + 2A_{26}\frac{\partial}{\partial y}\frac{\partial^2}{\partial x\,\partial y} + A_{66}\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2} + A_{26}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2} + \bar{A}_{22}\frac{\partial}{\partial y}\frac{\partial^2}{\partial y^2}.$$

(b) Nonlinear functions:

$$\begin{aligned}
K_{33}(w) = {} & \bar{B}_{11}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right] + \bar{B}_{22}\left[\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3}\right] + B_{16}\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial y} \\
& + 2B_{12}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + B_{26}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial y^3} + 3B_{16}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^2\,\partial y} + 3B_{26}\frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x\,\partial y^2} \\
& + 2B_{66}\left[\left(\frac{\partial^2 w}{\partial x\,\partial y^2}\right) + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] + (B_{12} + 2B_{66})\left(\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x\,\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\,\partial y}\right) \\
& + 4B_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\,\partial y} + 4B_{26}\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 w}{\partial y^2} - \left\{\frac{\partial}{\partial x}\left[\left(\bar{B}_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2} + 2B_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right)\frac{\partial w}{\partial x}\right.\right. \\
& \left. + \left(B_{16}\frac{\partial^2 w}{\partial x^2} + B_{26}\frac{\partial^2 w}{\partial y^2} + 2B_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[\left(B_{16}\frac{\partial^2 w}{\partial x^2} + B_{26}\frac{\partial^2 w}{\partial y^2} + 2B_{66}\frac{\partial w}{\partial x\,\partial y}\right)\frac{\partial w}{\partial x}\right. \\
& \left.\left. + \left(B_{12}\frac{\partial^2 w}{\partial x^2} + \bar{B}_{22}\frac{\partial^2 w}{\partial y^2} + 2B_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right)\frac{\partial w}{\partial y}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
K_{31}(u, w) = {} & \frac{\partial}{\partial x}\left[\left(\bar{A}_{11}\frac{\partial u}{\partial x} + A_{16}\frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial x} + \left(A_{16}\frac{\partial u}{\partial x} + A_{66}\frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial y}\right] \\
& + \frac{\partial}{\partial y}\left[\left(A_{16}\frac{\partial u}{\partial x} + A_{66}\frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial x} + \left(A_{12}\frac{\partial u}{\partial x} + A_{26}\frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial y}\right],
\end{aligned}$$

$$\begin{aligned}
K_{32}(v, w) = {} & \frac{\partial}{\partial x}\left[\left(A_{16}\frac{\partial v}{\partial x} + A_{12}\frac{\partial v}{\partial y}\right)\frac{\partial w}{\partial x} + \left(A_{66}\frac{\partial v}{\partial x} + A_{26}\frac{\partial v}{\partial y}\right)\frac{\partial w}{\partial y}\right] \\
& + \frac{\partial}{\partial y}\left[\left(A_{66}\frac{\partial v}{\partial x} + A_{26}\frac{\partial v}{\partial y}\right)\frac{\partial w}{\partial x} + \left(A_{26}\frac{\partial v}{\partial x} + \bar{A}_{22}\frac{\partial v}{\partial y}\right)\frac{\partial w}{\partial y}\right].
\end{aligned}$$

(2) *Cubic nonlinearity.*

$$\begin{aligned}
2N_{33}(w) = {} & \frac{\partial}{\partial x}\left\{\left[\bar{A}_{11}\left(\frac{\partial w}{\partial x}\right)^2 + A_{12}\left(\frac{\partial w}{\partial y}\right)^2 + 2A_{16}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial x} + \left[A_{16}\left(\frac{\partial w}{\partial x}\right)^2\right.\right. \\
& \left.\left. + A_{26}\left(\frac{\partial w}{\partial y}\right)^2 + 2A_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{\left[A_{16}\left(\frac{\partial w}{\partial x}\right)^2 + A_{26}\left(\frac{\partial w}{\partial y}\right)^2\right.\right. \\
& \left.\left. + 2A_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial x} + \left[A_{16}\left(\frac{\partial w}{\partial x}\right)^2 + \bar{A}_{22}\left(\frac{\partial w}{\partial y}\right)^2 + 2A_{26}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial y}\right\}.
\end{aligned}$$