Pergamon

Int. J. Mech. Sci. Vol. 36, No. 6, pp. 567–578, 1994

0020-7403/94 $7.00 + 0.00

0020-7403(94)E0006-5

ANALYTICAL MODELS OF SANDWICH PLATES WITH PIEZOELECTRIC STRIP-STIFFENERS

VICTOR BIRMAN
Engineering Education Center, University of Missouri-Rolla, 8001 Natural Bridge Road, St Louis, MO 63121, U.S.A.

(*Received* 4 *March* 1993; *and in revised form* 15 *November* 1993)

Abstract—A theory of sandwich plates with composite-material facings and piezoelectric strip-stiffeners bonded to the surface or embedded in the facings is developed. The stiffeners bonded to the surfaces are modeled using either the plane stress assumption or a first-order shear deformable theory. The former approach is appropriate if the stiffeners represent thin strips, while the latter method can be used in the case where the stiffeners are relatively deep. The stiffeners embedded in the facings in the form of piezoelectric strips are considered using the plane stress assumption.

NOTATION

A_i cross-sectional area of a stiffener ($i = r, s$)
A_{ij} extensional stiffnesses ($i, j = 1, 2, 6$)
D_i electric displacements ($i = x, y, z$)
D_{ij} bending stiffnesses ($i, j = 1, 2, 6$)
d_{ij} piezoelectric coefficients ($ij = 31, 33, 15$)
E modulus of elasticity
E_i electric field components ($i = x; y, z$)
F_i first moment of a stiffener with respect to the sandwich middle plane ($i = r, s$)
G shear modulus of piezoelectric material in the plane xy
G' shear modulus of piezoelectric material in the planes xz and yz
G_c shear modulus of a transversely isotropic core material in the planes xz and yz
G_{cx}, G_{cy} effective shear moduli of a honeycomb core in the planes xz and yz, respectively
h thickness of the core
h_1 thickness of a facing
h_p thickness of a piezoelectric strip
I_i moment of inertia of a stiffener with respect to the sandwich middle plane ($i = r, s$)
M_i stress couples ($i = x, y, xy$)
N_i in-plane stress resultants ($i = x, y, xy$)
Q_i transverse shear stress resultants ($i = x, y$)
Q_{ij} transformed reduced stiffnesses ($i, j = 1, 2, 6$)
s_{ij} elastic compliances ($ij = 11, 12, 13, 33, 44, 66$)
u, v in-plane displacements of the sandwich middle plane
w transverse deflections of the sandwich middle plane
x, y, z cartesian coordinates

Greek letters

ε strains
ε_{ii}^T dielectric permittivities ($ii = 11, 33$)
μ Poisson's ratio
ρ mass density
σ stresses
ϕ voltage in piezoelectric sensors
Ψ_x, Ψ_y bending slopes in the planes xz and yz, respectively

INTRODUCTION

Active control of thin-walled structures using piezoelectric sensors and actuators has been considered by a number of investigators. In particular, control of composite plates and shells using distributed piezoelectric actuators was a subject of research by Hagood *et al.* [1], Crawley and Lazarus [2], Tzou and Gadre [3]. Tzou and Zhong [4], Sung *et al.* [5], and Wang and Rogers [6]. Actuators considered in these papers were either included in the laminate as additional layers, or arranged in the form of patches bonded to the surface or embedded within the structure. A different approach to piezoelectric actuators was used by Birman [7, 8] who considered laminated plates reinforced by ribs that include piezoelectric strips. This approach can promise advantages similar to those enjoyed by reinforced structures as compared to unstiffened ones.

In the present paper, the theory of sandwich plates with piezoelectric strip-stiffeners is developed. The stiffeners can be either bonded to the surface of the sandwich or embedded within the facings. Sandwich plates considered in the paper have composite, generally laminated facings and a core with different transverse shear stiffnesses in the directions of the plate axes. This is a rather general model of sandwich plates that can be used to represent numerous applications. Mentioned here is a comprehensive review of Bert [9] that contains a discussion of existing research in sandwich plates with composite facings. The theory presented in the paper can be applied to the case of discrete as well as smeared stiffeners. The latter approach is feasible, if the stiffeners are identical and closely spaced.

FORMULATION OF THE PROBLEM AND BACKGROUND

Consider a sandwich panel with multilayered composite facings (Fig. 1). Piezoelectric stiffeners can be either bonded to the surface of the sandwich or embedded within the facings. The scheme with the stiffeners embedded in the core is not considered, since desirable effects can be achieved if the stiffeners are located as far from the middle surface as possible.

The analysis is based on the following sets of assumptions:

1. *Facings*
   - (i) Facings are identical and symmetric with respect to the middle surface;
   - (ii) plane stress state is dominant, i.e. transverse shearing and transverse normal stresses are negligible;
   - (iii) the behavior of the facing material is linear and elastic;
   - (iv) the thickness of the facings is constant throughout the sandwich.

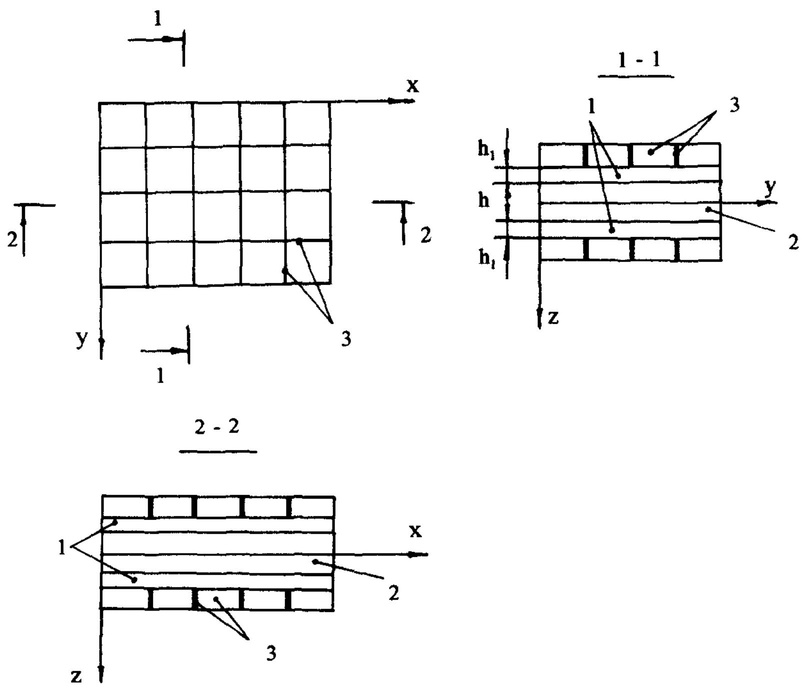

FIG. 1. Sandwich plate with piezoelectric stiffeners; 1 = composite facings, 2 = core, 3 = piezoelectric stiffeners.

2. *Core*
   (i) In-plane stresses are negligible;
   (ii) transverse normal stresses can be neglected;
   (iii) the behavior of the core material is linear and elastic;
   (iv) the thickness of the core is constant.

3. *Piezoelectric stiffeners*
   (i) The stiffeners are arranged in couples that include identical strips of a piezoelectric material, symmetric with respect to the middle surface;
   (ii) the material is macroscopically polarized by the poling process, so that the z-axis is the polar axis;
   (iii) the behavior of the material is linear and elastic;
   (iv) the thickness of each stiffener does not vary along its axis.

4. *General assumptions*
   (i) The theory developed in the paper is limited to geometrically linear deformations;
   (ii) perfect bonding is assumed between the core and the facings, and between the facings and the stiffeners;
   (iii) in-plane deformations are continuous throughout the core, facings and stiffeners.

Note that the core material is often assumed to be transversely isotropic. However, this assumption is wrong for a honeycomb core where the effective transverse shear stiffnesses in the planes xz and yz are different. An example of stiffnesses for a honeycomb core can be found in the book of Vinson and Sierakowski [10].

The assumptions for piezoelectric stiffeners do not specify the state of the stresses. Two cases considered here include the stiffeners in the state of plane stress and transversely shear deformable stiffeners with negligible transverse normal stresses.

A piezoelectric material polarized along the z-axis remains macroscopically isotropic in the x-y plane. Accordingly, the piezoelectric constitutive relations for such a material are [11]:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ D_z \\ \varepsilon_{yz} \\ D_y \\ \varepsilon_{xz} \\ D_x \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & d_{31} & & & & & \\ s_{12} & s_{11} & s_{13} & d_{31} & & & & & \\ s_{13} & s_{13} & s_{33} & d_{33} & & & & & \\ d_{31} & d_{31} & d_{33} & \varepsilon_{33}^T & & & & & \\ & & & & s_{44} & d_{15} & & & \\ & & & & d_{15} & \varepsilon_{11}^T & & & \\ & & & & & & s_{44} & d_{15} & \\ & & & & & & d_{15} & \varepsilon_{11}^T & \\ & & & & & & & & s_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ E_z \\ \sigma_{yz} \\ E_y \\ \sigma_{xz} \\ E_x \\ \sigma_{xy} \end{Bmatrix} \tag{1}$$

where σ_i, σ_{ij}, ε_i and ε_{ij} denote axial and shearing stresses and strains, respectively. Unoccupied positions in the matrix in the right side of Eqns (1) correspond to zeros. Numerous piezoelectric ceramics, e.g. barium titanate, lead zirconate titanate, lead metaniobate, and others are characterized by the constitutive Eqns (1).

In the state of plane stress, the strain–stress equations obtained from relations (1) are:

$$\begin{aligned} \varepsilon_{xy} &= s_{66}\sigma_{xy} \\ \varepsilon_x &= s_{11}\sigma_x + s_{12}\sigma_y + d_{31}E_z \\ \varepsilon_y &= s_{12}\sigma_x + s_{11}\sigma_y + d_{31}E_z, \end{aligned} \tag{2}$$

where:

$$s_{11} = E^{-1} \quad s_{12} = -\mu E^{-1} \quad s_{66} = G^{-1}, \tag{3}$$

E, G and μ are the elasticity and shear moduli, and the Poisson's ratio of the material in the plane of transverse isotropy (xy plane).

If transverse shear deformability is included in the analysis, Eqns (2) must be complemented by the expressions for ε_{yz} and ε_{xz}. Limiting the analysis to the case where the only nonzero electric field component is acting in the thickness direction, one obtains:

$$\varepsilon_{yz} = G'^{-1}\sigma_{yz} \quad \varepsilon_{xz} = G'^{-1}\sigma_{xz}, \tag{4}$$

where G' is the shear modulus in the planes xz and yz.

Two approaches to piezoelectric stiffeners are considered, i.e. discrete stiffeners and smeared stiffeners. The latter approach, i.e. smeared stiffeners technique, is justified in the case of identical closely spaced stiffeners. To illustrate the difference between discrete and smeared stiffeners, consider a system of stiffeners oriented in the y-direction. If the coordinates of the centroids of the stiffeners are denoted by x_r and $f(x)$ is a contribution of the stiffeners to the stress resultant or to the stress couple, the discrete stiffeners approach in conjunction with Galerkin's procedure yields:

$$F_1 = \int_0^a \delta(x - x_r) f(x)\,\mathrm{d}x = \sum_r f(x_r), \tag{5}$$

where $\delta(x - x_r)$ is Dirac's delta function.

According to the smeared stiffeners technique,

$$\delta(x - x_r) = l_r^{-1}, \tag{6}$$

where l_r is the spacing of the stiffeners, i.e. the contribution is given by:

$$F_2 = \frac{1}{l_r}\int_0^a f(x)\,\mathrm{d}x. \tag{7}$$

Obviously, the smeared stiffeners technique that yields simpler analytical solutions can be used if F_1 is close to F_2. For example, if $f(x) = \sin\frac{\pi x}{l}$ and $x_r = \frac{a}{4}, \frac{a}{2}$ and $\frac{3a}{4}$, $F_1 = 2.414$ and $F_2 = 2.546$, i.e. the error associated with the smeared stiffeners technique is 5.5%. However, if the number of stiffeners is reduced to two, i.e. $x_r = 0.333a$ and $0.667a$, $F_1 = 1.732$ and $F_2 = 1.910$, so that the error increases to 10.3%. In the case of one stiffener located at $x_r = \frac{a}{2}$ the error is equal to 27.3%. This explains why the smeared stiffeners technique should be applied with caution if the number of stiffeners is small.

GOVERNING EQUATIONS

According to the assumption of continuous in-plane deformations, displacements in the x and y directions at a point located at a distance z from the middle plane are:

$$\begin{aligned} U &= u(x, y) + z\Psi_x(x, y) \\ V &= v(x, y) + z\Psi_y(x, y), \end{aligned} \tag{8}$$

respectively.

The linear mechanical strain–displacement relationships corresponding to the first-order shear deformation theory tacitly introduced through Eqns (8) are:

$$\begin{aligned} \varepsilon_x &= u_{,x} + z\Psi_{x,x} \\ \varepsilon_y &= v_{,y} + z\Psi_{y,y} \\ \varepsilon_{xy} &= u_{,y} + v_{,x} + z(\Psi_{x,y} + \Psi_{y,x}) \\ \varepsilon_{yz} &= \Psi_y + w_{,y} \\ \varepsilon_{xz} &= \Psi_x + w_{,x} \\ \varepsilon_z &= 0. \end{aligned} \tag{9}$$

The constitutive relations for a honeycomb core material are:

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} G_{cy} & 0 \\ 0 & G_{cx} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}. \tag{10}$$

If the core is transversely isotropic, $G_{cx} = G_{cy} = G_c$.

The stress–strain equations for the k-th layer in the facings read

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ & Q_{22} & Q_{26} \\ \text{sym} & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}. \tag{11}$$

The stress–strain equations for piezoelectric stiffeners can be obtained from the inverse of Eqns (2) and (4):

$$\begin{aligned} \sigma_x &= \frac{E}{1-\mu^2}[\varepsilon_x + \mu\varepsilon_y - (1+\mu)d_{31}E_z] \\ \sigma_y &= \frac{E}{1-\mu^2}[\varepsilon_y + \mu\varepsilon_x - (1+\mu)d_{31}E_z] \\ \sigma_{xy} &= G\varepsilon_{xy}, \end{aligned} \tag{12}$$

and:

$$\{\sigma_{yz}, \sigma_{xz}\} = G'\{\varepsilon_{yz}, \varepsilon_{xz}\}, \tag{13}$$

respectively. The latter equations are needed only if the stiffeners are shear deformable.

Note that if a stiffener is bonded to the outer surface of the facing rather than embedded within it, the stresses that have to be accounted for can be reduced to normal stresses in the stiffener axis direction and transverse shearing stresses (in the case of a shear deformable stiffener). The corresponding stress–strain equations can be obtained from Eqns (12) and (13), where $\mu = 0$. On the other hand, if the stiffener is embedded within the facings, in-plane normal stresses in the direction perpendicular to the stiffener axis and in-plane shearing stresses have to be included in the analysis. This reflects the continuity of in-plane displacements that is attributed to perfect bonding between the stiffener and the facing. Torsional stiffness of strip-stiffeners and associated stresses are neglected in this paper.

The stress resultants and the stress couples are defined in the customary manner by integrating the corresponding stresses and the moments of these stresses about the middle plane through the thickness of the structure. They can be represented as follows:

$$\begin{aligned} N_i &= N_i' + N_i'' \\ M_i &= M_i' + M_i'' \\ Q_j &= Q_j' + Q_j'', \end{aligned} \tag{14}$$

where $i = x, y$ and $xy, j = yz$ and xz, the quantities with a prime indicate the contributions of the facings and the core, and those with double prime refer to the contribution of piezoelectric stiffeners. Note that Q_j'' appears only if the stiffeners are shear deformable.

The three types of stiffeners included in the analysis are:

- stiffeners bonded to outer surfaces of the facings and working in a state of plane stress (plane–stress stiffeners);
- shear deformable stiffeners bonded to outer surfaces of the facings;
- plane–stress stiffeners embedded within the facings (Fig. 2).

Consider contributions of the core, stiffeners and facings to the stress couples and stress resultants.

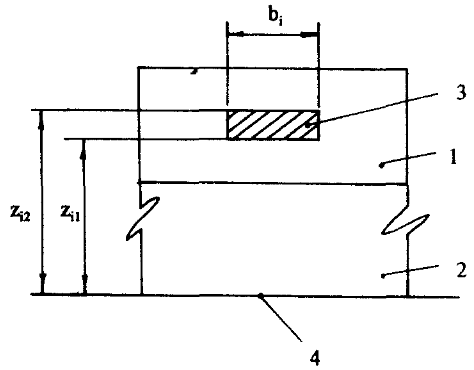

FIG. 2. Cross-section of a plate with the piezoelectric stiffener; 1 = facing, 2 = core, 3 = cross-section of the stiffener, 4 = middle surface of sandwich. The width of the stiffener is b_i, the height is $(z_{iz} - z_{i1})$.

1. *Contributions of the core* (Q'_j)
By definition,

$$Q'_j = \int_{-h/2}^{h/2} \sigma_j \, dz = G_c \varepsilon_j h. \tag{15}$$

Using notation of the theory of composite and sandwich plates, one can write

$$\begin{Bmatrix} Q'_y \\ Q'_x \end{Bmatrix} = \begin{bmatrix} A^c_{44} & 0 \\ 0 & A^c_{55} \end{bmatrix} \begin{Bmatrix} \Psi_y + w_{,y} \\ \Psi_x + w_{,x} \end{Bmatrix}, \tag{16}$$

where $A^c_{44} = G_{cy}h$, $A^c_{55} = G_{cx}h$.
2. *Contributions of shear deformable stiffeners to transverse shear stress resultants* (Q''_j)

$$Q''_y = \sum_r \int_{2A_r} \delta(x - x_r)\sigma_{yz}\, d(2A_r)$$

$$Q''_x = \sum_s \int_{2A_s} \delta(y - y_s)\sigma_{xz}\, d(2A_s). \tag{17}$$

Coefficient 2 in Eqns (17) accounts for two stiffeners, each of them with the cross-sectional area A_s or A_r that are symmetric about the middle plane. Accordingly,

$$Q''_y = 2\sum_r \delta(x - x_r)\, G' A_r(\Psi_y + w_{,y})$$

$$Q''_x = 2\sum_s \delta(y - y_s)\, G' A_s(\Psi_x + w_{,x}). \tag{18}$$

3. *Contributions of plane–stress stiffeners bonded to outer surfaces of the facings* (N''_i, M''_i)
The stress resultant N''_x and the stress couple M''_x in the stiffeners oriented along the x-axis are:

$$\{N''_x, M''_x\} = \sum_s \int_{2A_s} \delta(y - y_s)\sigma_x\{1, z\}\, d(2A_s). \tag{19}$$

Transformations yield:

$$N''_x = 2\sum_s \delta(y - y_s) E\left[A_s u_{,x} - \frac{1}{2} d_{31} \int_{2A_s} E_z\, d(2A_s)\right]$$

$$M''_x = 2\sum_s \delta(y - y_s) E\left[I_s \Psi_{x,x} - \frac{1}{2} d_{31} \int_{2A_s} z E_z\, d(2A_s)\right]. \tag{20}$$

The integrals in the right side of Eqns (20) account for the possibility of different electric fields applied to the stiffeners bonded to the upper and lower facings. If these fields are identical, i.e. $E_z = E_z(x, y)$ is the same in the stiffeners on the opposite surfaces of the sandwich (in-phase voltages),

$$N''_x = 2\sum_s \delta(y - y_s) E A_s (u_{,x} - d_{31} E_z)$$
$$M''_x = 2\sum_s \delta(y - y_s) E I_s \Psi_{x,x}. \tag{21}$$

In this case the piezoelectric effect affects the stress resultant while its contribution to the stress couple is equal to zero. On the other hand, if electric fields of opposite signs are applied to the stiffeners symmetric with respect to the middle plane of the sandwich (out-of-phase voltage),

$$N''_x = 2\sum_s \delta(y - y_s) E A_s u_{,x}$$
$$M''_x = 2\sum_s \delta(y - y_s) E (I_s \Psi_{x,x} - d_{31} E_z F_s). \tag{22}$$

In Eqns (22), the piezoelectrically induced stress resultant is equal to zero, but the piezoelectric stress couple is present.

Note that a comparison of the effectiveness of active control of orthotropic plates using piezoelectric stiffeners bonded to the opposite surfaces and in-phase or out-of-phase voltages was presented by Birman and Adali [12]. It was shown that out-of-phase voltages, i.e. piezoelectric stress couples, are much more effective for active control of transverse (out-of-plane) motion than in-phase voltages and associated stress resultants.

The contributions of the stiffeners oriented in the y-directions are:

$$N''_y = 2\sum_r \delta(x - x_r) E\left[A_r v_{,y} - \frac{1}{2} d_{31} \int_{2A_r} E_z \, \mathrm{d}(2A_r)\right]$$
$$M''_y = 2\sum_r \delta(x - x_r) E\left[I_r \Psi_{y,y} - \frac{1}{2} d_{31} \int_{2A_r} z E_z \, \mathrm{d}(2A_r)\right]. \tag{23}$$

4. *Contributions of the facings* (*the case where the stiffeners are bonded to the outer surfaces*)

The customary approach yields:

$$\begin{Bmatrix} N'_x \\ N'_y \\ N'_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ \text{sym} & & A_{66} \end{bmatrix} \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix}$$

$$\begin{Bmatrix} M'_x \\ M'_y \\ N'_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ \text{sym} & & D_{66} \end{bmatrix} \begin{Bmatrix} \Psi_{x,x} \\ \Psi_{y,y} \\ \Psi_{x,y} + \Psi_{y,x} \end{Bmatrix}, \tag{24}$$

where:

$$A_{ij} = 2\int_{h/2}^{h/2 + h_1} Q_{ij} \, \mathrm{d}z$$
$$D_{ij} = 2\int_{h/2}^{h/2 + h_1} Q_{ij} z^2 \, \mathrm{d}z. \tag{25}$$

5. *Contributions of plane–stress piezoelectric stiffeners embedded in the facings* (N''_i, M''_i)

In this case, the stiffeners act as parts of the facings and their stress–strain relationships are given by Eqns (12). For the stiffeners oriented in the x-direction, one obtains:

$$N''_x = 2\sum_s \delta(y - y_s) \tilde{E}\left[A_s (u_{,x} + \mu v_{,y}) - \frac{1}{2}(1 + \mu) d_{31} \int_{2A_s} E_z \, \mathrm{d}(2A_s)\right]$$
$$N''_y = 2\sum_s \delta(y - y_s) \tilde{E}\left[A_s (v_{,y} + \mu u_{,x}) - \frac{1}{2}(1 + \mu) d_{31} \int_{2A_s} E_z \, \mathrm{d}(2A_s)\right]$$

$$N''_{xy} = 2\sum_s \delta(y - y_s) G A_s (u_{,y} + v_{,x})$$

$$M''_x = 2\sum_s \delta(y - y_s)\tilde{E}\left[I_s(\Psi_{x,x} + \mu\Psi_{y,y}) - \frac{1}{2}(1+\mu) d_{31} \int_{2A_s} z E_z \, \mathrm{d}(2A_s)\right]$$

$$M''_y = 2\sum_s \delta(y - y_s)\tilde{E}\left[I_s(\Psi_{y,y} + \mu\Psi_{x,x}) - \frac{1}{2}(1+\mu) d_{31} \int_{2A_s} z E_z \, \mathrm{d}(2A_s)\right]$$

$$M''_{xy} = \frac{1}{2}\sum_s \delta(y - y_s) G I_s (\Psi_{x,y} + \Psi_{y,x}), \tag{26}$$

where $\tilde{E} = E/(1 - \mu^2)$.

Note that Eqns (26) are obtained by assumption that embedded piezoelectric stiffeners are perfectly bonded to the facings so that deformations in the stiffeners and in the facings are continuous.

If each couple of stiffeners on both sides of the middle plane is subjected to in-phase electric fields, $\int_{2A_s} E_z \, \mathrm{d}(2A_s) = 2E_z A_s$ and $\int_{2A_s} z E_z \, \mathrm{d}(2A_s) = 0$ in Eqns (26). In the case of out-of-phase electric fields, $\int_{2A_s} E_z \, \mathrm{d}(2A_s) = 0$ and $\int_{2A_s} z E_z \, \mathrm{d}(2A_s) = 2E_z F_s$.

The contributions of the stiffeners oriented in the y-direction can be obtained from Eqns (26) by replacing $\delta(y - y_s)$ with $\delta(x - x_r)$, and A_s and I_s with A_r and I_r, respectively.

6. *Contributions of the facings that contain embedded piezoelectric stiffeners* (N'_i, M'_i)

Eqns (24) must be replaced with equations that reflect the absence of the facing material in the regions occupied by the stiffeners (Fig. 2). Accordingly, the stress resultants and couples given by Eqns (24) must be reduced by:

$$\{\Delta N'_i, \Delta M'_i\} = \sum_r \delta(x - x_r) b_r \left[\int_{z_{r1}}^{z_{r2}} \sigma_i\{1, z\}\,\mathrm{d}z + \int_{-z_{r2}}^{-z_{r1}} \sigma_i\{1, z\}\,\mathrm{d}z \right] + \sum_s \delta(y - y_s) b_s \left[\int_{z_{s1}}^{z_{s2}} \sigma_i\{1, z\}\,\mathrm{d}z + \int_{-z_{s2}}^{-z_{s1}} \sigma_i\{1, z\}\,\mathrm{d}z \right], \tag{27}$$

where b_r and b_s are the widths of the respective stiffeners, and positive numbers z_{r1}, z_{r2}, z_{s1}, z_{s2} are the coordinates of the interfaces between the piezoelectric stiffeners and the facings (Fig. 2).

As a result of subtracting Eqns (27) from Eqns (24), extensional and bending stiffnesses in the latter equations must be modified, i.e. $A_{ij} \to A'_{ij}$ and $D_{ij} \to D'_{ij}$, where:

$$A'_{ij} = A_{ij} - 2\left[\sum_r \delta(x - x_r) b_r \int_{z_{r1}}^{z_{r2}} Q_{ij}\,\mathrm{d}z + \sum_s \delta(y - y_s) b_s \int_{z_{s1}}^{z_{s2}} Q_{ij}\,\mathrm{d}z \right]$$

$$D'_{ij} = D_{ij} - 2\left[\sum_r \delta(x - x_r) b_r \int_{z_{r1}}^{z_{r2}} Q_{ij} z^2\,\mathrm{d}z + \sum_s \delta(y - y_s) b_s \int_{z_{s1}}^{z_{s2}} Q_{ij} z^2\,\mathrm{d}z \right]. \tag{28}$$

If the smeared stiffeners technique is applied, Eqns (28) are modified accordingly, i.e. $\sum_r \delta(x - x_r) \to l_r^{-1}$, $\sum_s \delta(y - y_s) \to l_s^{-1}$.

7. *Summary of stress resultants and stress couples*

Consider expressions for total stress resultants and stress couples for all cases discussed in the paper.

Case A: plane–stress stiffeners bonded to the outer surfaces of the facings. Combining Eqns (16), (20), (23) and (24) one obtains:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} - \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} \tag{29}$$

$$\{Q\} = [G_c] h \{\gamma\}, \tag{30}$$

where N, M and Q are the vectors of in-plane stress resultants, stress couples and transverse shear stress resultants, respectively. The vectors of middle plane strains and the changes of curvature and twist are defined as:

$$\varepsilon^0 = \{u_{,x} \ v_{,y} \ u_{,y} + v_{,x}\}^T$$
$$\kappa = \{\Psi_{x,x} \ \Psi_{y,y} \ \Psi_{x,y} + \Psi_{y,x}\}^T, \tag{31}$$

and the vector of transverse shear strains is:

$$\gamma = \{\varepsilon_{yz} \ \varepsilon_{xz}\}^T. \tag{32}$$

The matrices $\bar{A}$ and $\bar{D}$ are presented in the Appendix. The matrix $[G_c]$ is given in the right side of Eqn (10). The components of the vectors of piezoelectric terms in Eqn (29) are:

$$\{N_{px}, M_{px}\} = d_{31} E \sum_s \delta(y - y_s) \int_{2A_s} E_z\{1, z\}\,\mathrm{d}(2A_s)$$
$$\{N_{py}, M_{py}\} = d_{31} E \sum_r \delta(x - x_r) \int_{2A_r} E_z\{1, z\}\,\mathrm{d}(2A_r)$$
$$\{N_{pxy}, M_{pxy}\} = 0. \tag{33}$$

Case B: shear deformable stiffeners bonded to the outer surfaces of the facings. Eqns (29) remain valid. However, Eqns (30) must be replaced with:

$$\{Q\} = [\tilde{A}]\{\gamma\}, \tag{34}$$

where $[\tilde{A}]$ is a diagonal matrix of the second order whose elements are given in the Appendix.

Case C: plane–stress piezoelectric stiffeners embedded in the facings. In this case, Eqns (16), (24) and (26) must be combined. The stiffnesses A_{ij} and D_{ij} in Eqns (24) must be replaced with A'_{ij} and D'_{ij} defined by Eqns (28). The result is similar to Eqns (29) and (30), although the matrices $\bar{A}$ and $\bar{D}$, as well as piezoelectric terms in N_p and M_p are different (see Appendix).

8. *Equations of motion*

Equations written in terms of stress resultants and stress couples correspond to the first-order shear deformation theory [13]:

$$N_{x,x} + N_{xy,y} = I_1\ddot{u} + I_2\ddot{\Psi}_x$$
$$N_{xy,x} + N_{y,y} = I_1\ddot{v} + I_2\ddot{\Psi}_y$$
$$Q_{x,x} + Q_{y,y} + \underline{(N_x w_{,x} + N_{xy} w_{,y})_{,x} + (N_{xy} w_{,x} + N_y w_{,y})_{,y}} = q + I_1\ddot{w}$$
$$M_{x,x} + M_{xy,y} - Q_x = I_3\ddot{\Psi}_x + I_2\ddot{u}$$
$$M_{xy,x} + M_{y,y} - Q_y = I_3\ddot{\Psi}_y + I_2\ddot{v}, \tag{35}$$

where q is a transverse load (in the z-direction), and I_1, I_2 and I_3 are normal, coupled normal-rotary, and rotary inertia coefficients, respectively, given by:

$$\{I_1, I_2, I_3\} \doteq \int_z \{1, z, z^2\}\,\rho\,\mathrm{d}z. \tag{36}$$

Note that nonlinear terms in the third Eqn (35) must be omitted. Therefore, underlined terms in the left side of this equation should read:

$$-[(N_{px} w_{,x})_{,x} + (N_{py} w_{,y})_{,y}] + \bar{N}_x w_{,xx} + 2\bar{N}_{xy} w_{,xy} + \bar{N}_y w_{,yy},$$

where the quantities with an overbar indicate applied in-plane loads.

If the stiffeners are bonded to the outer surfaces of the facings,

$$I_1 = \rho_c h + 2\rho_f h_1 + 2\sum_r \delta(x - x_r)\rho_p A_r + 2\sum_s \delta(y - y_s)\rho_p A_s$$
$$I_2 = 0$$
$$I_3 = \frac{\rho_c h^3}{12} + \frac{2}{3}\rho_f\left[\left(h_1 + \frac{h}{2}\right)^3 - \left(\frac{h}{2}\right)^3\right]$$
$$+ 2\sum_r \delta(x - x_r)\rho_p I_r + 2\sum_s \delta(y - y_s)\rho_p I_s, \tag{37}$$

where ρ_c, ρ_f, and ρ_p are mass densities of the core, facings and piezoelectric materials, respectively. The coupled inertia coefficient, I_2, is equal to zero because of the symmetry of the structure about the middle plane. If the stiffeners are embedded in the facings, the inertia coefficients become:

$$I_1 = \rho_c h + 2\rho_f h_1 + 2\sum_r \delta(x - x_r)(\rho_p - \rho_f)A_r + 2\sum_s \delta(y - y_s)(\rho_p - \rho_f)A_s$$

$$I_2 = 0$$

$$I_3 = \frac{\rho_c h^3}{12} + \frac{2}{3}\rho_f\left[\left(h_1 + \frac{h}{2}\right)^3 - \left(\frac{h}{2}\right)^3\right]$$

$$+ 2\sum_r \delta(x - x_r)(\rho_p - \rho_f)I_r + 2\sum_s \delta(y - y_s)(\rho_p - \rho_f)I_s. \tag{38}$$

The substitution of Eqns (29) and (30) or (34) into equations of motion (35) yields equations of motion in terms of generalized displacements that are similar to those presented by Reddy [13] for shear deformable composite plates:

$$[L]\{\Delta\} = \{f\}, \tag{39}$$

where

$$\{\Delta\} = \{u, v, w, \Psi_x, \Psi_y\}^T$$

$$\{f\} = \{N_{px,x}, N_{py,y}, q, M_{px,x}, M_{py,y}\}^T, \tag{40}$$

and $[L]$ is a symmetric matrix of differential operators with the following elements:

$$\mathscr{L}_{11} = A_{11}d_{11} + 2A_{16}d_{12} + A_{66}d_{22} - I_1 d_{tt}$$

$$\mathscr{L}_{13} = \mathscr{L}_{14} = \mathscr{L}_{15} = \mathscr{L}_{23} = \mathscr{L}_{24} = \mathscr{L}_{25} = 0$$

$$\mathscr{L}_{22} = 2A_{26}d_{12} + A_{22}d_{22} + A_{66}d_{11} - I_1 d_{tt}$$

$$\mathscr{L}_{33} = -A_{55}d_{11} - A_{44}d_{22} - I_1 d_{tt} - \bar{N}_x d_{11} - 2\bar{N}_{xy}d_{12} - \bar{N}_y d_{22}$$

$$+ N_{px,1}d_1 + N_{px}d_{11} + N_{py,2}d_2 + N_{py}d_{22}$$

$$\mathscr{L}_{34} = A^c_{55}d_1$$

$$\mathscr{L}_{35} = -A^c_{44}d_2$$

$$\mathscr{L}_{44} = D_{11}d_{11} + 2D_{16}d_{12} + D_{66}d_{22} - A_{55} - I_3 d_{tt}$$

$$\mathscr{L}_{45} = (D_{12} + D_{66})d_{12} + D_{16}d_{11} + D_{26}d_{22}$$

$$\mathscr{L}_{55} = 2D_{26}d_{12} + D_{22}d_{22} + D_{66}d_{11} - A_{44} - I_3 d_{tt}. \tag{41}$$

In Eqns (41), $d_{ij} = \partial^2/\partial x_i \partial x_j$ $(x_1 = x,\ x_2 = y)$, $d_{tt} = \partial^2/\partial t^2$.

Equations of motion (39) must be complemented by boundary conditions. Kinematic conditions in terms of displacements and bending slopes can be immediately formulated. Static conditions are given in terms of stress resultants and stress couples introduced by Eqns (14). Note that if piezoelectric stiffeners are not extended to the boundaries, N_i'' M_i'' and Q_i'' in the boundary conditions are equal to zero.

The system of Eqns (39) can be used for the analysis both in the case where piezoelectric strips work in the sensor mode, as well as if they are used as actuators. In the former case, the response can be considered by assumption that voltages generated in the sensors as a result of deformations are too small to affect the behavior of the structure. This assumption that has been proven for various structures and loading regimes is tacitly accepted in almost all studies of piezoelectric sensors.

Accordingly, the response of the sandwich plate can be found from Eqns (39) without piezoelectric terms. Then the electric field and the corresponding voltage in piezoelectric sensors can be obtained using their constitutive equations. For example, for the strips bonded to the surface of the plate and oriented in the x-direction the first and the fourth Eqns (1) yield:

$$E_z = \frac{d_{31}E\varepsilon_x}{d_{31}^2 E - \varepsilon_{33}^T}. \tag{42}$$

The voltage generated in the sensor can be obtained as shown in Ref. [14]. In the case where terminals (connecting points) on the sensor are closely spaced,

$$\phi = -\frac{d_{31}E}{d_{31}^2 E - \varepsilon_{33}^T}\int_{h_p} \varepsilon_x \, dz. \tag{43}$$

If the piezoelectric strips are used as actuators, the response of the plate can be calculated from Eqns (39) where piezoelectric terms are treated as loading terms. In some situations certain piezoelectric strips can be used as actuators, while other strips work in the sensor regime. If actuators and sensors work in couples, i.e. each couple of piezoelectric strips that are symmetric about the middle surface includes either sensors or actuators, the present theory can be used without modifications. If each couple of piezoelectric elements includes one sensor and one actuator, the theory can be modified to include this case. However, this modification is straightforward because it is confined to the loading piezoelectric terms.

CONCLUSIONS

A comprehensive theory of sandwich plates with piezoelectric strip-stiffeners bonded to the surface or embedded in the facings is presented. The layers of composite facings can be arbitrarily oriented. The core can have different transverse shear stiffnesses in the directions of the plate axes, as is the case for a honeycomb core. The formulation for a transversely isotropic core is obtained as a particular case. Embedded piezoelectric stiffeners are modeled using the plane stress assumption. The stiffeners bonded to the surface of the sandwich can also be assumed in the state of plane stress. An additional model considered for the stiffeners bonded to the surfaces of the sandwich incorporates transverse shear deformability. The paper presents governing equations of motion and the boundary conditions. These equations can be used both in the case where the stiffeners are used as sensors, as well as if they work as actuators.

Acknowledgements—This research was supported by the Army Research Office grant DAAL 03-92-G0301. The author would like to acknowledge the assistance of Dr Gary L. Anderson, Member of the Engineering and Environmental Sciences Division of the Army Research Office. Discussions with Professor Horn-Sen Tzou of the University of Kentucky are also warmly appreciated.

REFERENCES

1. N. W. Hagood, E. F. Crawley, J. de Luis and E. H. Anderson, Development of integrated components for control of intelligent systems, in *Smart Materials, Structures and Mathematical Issues* (edited by C. A. Rogers) p. 80. Technomic, Lancaster (1988).
2. E. F. Crawley and K. B. Lazarus, Induced strain actuation of isotropic and anisotropic plates, in *Proceedings of the* 30*th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, p. 1451 (1989).
3. H. S. Tzou and M. Gadre, Theoretical analysis of a multilayered thin shell coupled with piezoelectric shell actuators for distributed vibration controls. *J. Sound Vibr.* **132**, 433 (1989).
4. H. S. Tzou and J. P. Zhong, Theory of hexagonal symmetrical piezoelectric thick shells applied to smart structures, in *Structural Vibration and Acoustics* (edited by T. C. Huang *et al.*) p. 7. ASME, DE-Vol. 34, New York (1991).
5. C. S. Sung, C. F. Chen and S. G. Chen, Piezoelectric modal sensor/actuator design for monitoring/generating flexural and torsional vibrations of cylindrical shells, in *Structural Vibration and Acoustics* (edited by T. C. Huang *et al.*) p. 17. ASME, DE-Vol. 34, New York (1991).
6. B.-T. Wang and C. A. Rogers, Laminate plate theory for spatially distributed induced strain actuators. *J. Composite Mater.* **25**, 433 (1991).
7. V. Birman, Theory of geometrically nonlinear composite plates with piezoelectric stiffeners, in *Active Control of Noise and Vibration* 1992 (edited by C. J. Radcliffe, K.-W. Wang, H. S. Tzou and E. W. Hendricks) p. 231. ASME, DSC-Vol. 38, New York (1992).
8. V. Birman, Active control of composite plates using piezoelectric stiffeners. *Int. J. Mech. Sci.* **35**, 387 (1993).
9. C. W. Bert, Shear deformation and sandwich configuration, in *Buckling and Postbuckling of Composite Plates* (edited G. J. Turvey and I. H. Marshall). Elsevier, London (in press).
10. J. R. Vinson and R. L. Sierakowski, in *The Behavior of Structures Composed of Composite Materials*, p. 107. Martinus Nijhoff, Dordrecht (1986).
11. W. A. Smith, The key design principle for piezoelectric ceramic/polymer composites, in *Recent Advances in Adaptive and Sensory Materials and Their Applications* (edited by C. A. Rogers and R. C. Rogers) p. 825. Technomic, Lancaster (1992).

12. V. BIRMAN and S. ADALI, Active optimum control of orthotropic plates using piezoelectric stiffeners, in *Intelligent Structures, Materials, and Vibrations* (edited by M. SHAHINPOOR and H. S. TZOU), p. 43, ASME, DE-Vol. 58, New York (1993).
13. J. N. REDDY, in *Energy and Variational Methods in Applied Mechanics*, p. 390. Wiley, New York (1984).
14. D. A. SARAVANOS, V. BIRMAN and D. A. HOPKINS, Detection of delaminations in composite beams using piezoelectric sensors, in *Proceedings of the 35th Structures, Structural Dynamics, and Materials Conference* (1994) (in press).

APPENDIX

Elements of the matrices $\bar{A}$ and $\bar{D}$ in Eqns (29) for case A (plane–stress stiffeners bonded to the outer surfaces of the facings):

$$\bar{A}_{11} = A_{11} + 2\sum_s \delta(y - y_s) E A_s$$

$$\bar{A}_{22} = A_{22} + 2\sum_r \delta(x - x_r) E A_r$$

$$\bar{A}_{ij} = A_{ij}, \quad \text{if } ij \neq 11 \text{ or } 22$$

$$\bar{D}_{11} = D_{11} + 2\sum_s \delta(y - y_s) E I_s$$

$$\bar{D}_{22} = D_{22} + 2\sum_r \delta(x - x_r) E I_r$$

$$\bar{D}_{ij} = D_{ij}. \quad \text{if } ij \neq 11 \text{ or } 22.$$

Elements of the matrix $\tilde{A}$ in Eqn (34) for case B (shear-deformable stiffeners bonded to the outer surfaces of the facings):

$$\tilde{A}_{44} = A^c_{44} + 2\sum_r \delta(x - x_r) G' A_r$$

$$\tilde{A}_{55} = A^c_{55} + 2\sum_s \delta(y - y_s) G' A_s$$

$$\tilde{A}_{45} = \tilde{A}_{54} = 0.$$

Elements of matrices $\bar{A}$ and $\bar{D}$ in Eqns (29) for case C (plane–stress stiffeners embedded in the facings):

$$\bar{A}_{11} = A'_{11} + 2\sum_s \delta(y - y_s) \tilde{E} A_s + 2\sum_r \delta(x - x_r) \tilde{E} A_r$$

$$\bar{A}_{12} = A'_{12} + 2\sum_s \delta(y - y_s) \mu \tilde{E} A_s + 2\sum_r \delta(x - x_r) \mu \tilde{E} A_r$$

$$\bar{A}_{22} = A'_{22} + 2\sum_s \delta(y - y_s) \tilde{E} A_s + 2\sum_r \delta(x - x_r) \tilde{E} A_r$$

$$\bar{A}_{66} = A'_{66} + 2\sum_s \delta(y - y_s) G A_s + 2\sum_r \delta(x - x_r) G A_r$$

$$\bar{A}_{21} = \bar{A}_{12}$$

$$\bar{A}_{16} = A'_{16} \quad \bar{A}_{26} = A'_{26}$$

$$\bar{D}_{11} = D'_{11} + 2\sum_s \delta(y - y_s) \tilde{E} I_s + 2\sum_r \delta(x - x_r) \tilde{E} I_r$$

$$\bar{D}_{21} = \bar{D}_{12}$$

$$\bar{D}_{12} = D'_{12} + 2\sum_s \delta(y - y_s) \mu \tilde{E} I_s + 2\sum_r \delta(x - x_r) \mu \tilde{E} I_r$$

$$\bar{D}_{22} = D'_{22} + 2\sum_s \delta(y - y_s) \tilde{E} I_s + 2\sum_r \delta(x - x_r) \tilde{E} I_r$$

$$\bar{D}_{66} = D'_{66} + 2\sum_s \delta(y - y_s) G I_s + 2\sum_r \delta(x - x_r) G I_r$$

$$\bar{D}_{16} = D'_{16} \quad \bar{D}_{26} = D'_{26}.$$

Components of the vectors of piezoelectric terms in Eqns (29) for case C:

$$\{N_{px}, M_{px}\} = d_{31} \tilde{E}(1 + \mu) \left[\sum_s \delta(y - y_s) \int_{2A_s} E_z \{1, z\} \, \mathrm{d}(2A_s) + \sum_r \delta(x - x_r) \int_{2A_r} E_z \{1, z\} \, \mathrm{d}(2A_r) \right]$$

$$N_{py} = N_{px} \quad M_{py} = M_{px}$$

$$\{N_{pxy}, M_{pxy}\} = 0.$$