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Problems of Dynamic Buckling of Antisymmetric Rectangular Laminates*

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ABSTRACT

Dynamic buckling of antisymmetrically laminated angle-ply rectangular plates due to axial loads proportional to time and axial step loads is considered. The nonlinear response of initially imperfect plates is determined from the numerical solution of the governing differential equation. In the case of the step loading this equation can be solved analytically.

In the particular case of a perfect plate the solution of the linear problem yields the condition of dynamic buckling. Another problem considered in the paper is the behavior of an imperfect plate initially loaded by axial static stresses. The static response is determined first and the motion of the plate is superimposed on the static displacements in the second phase of the solution.

1 INTRODUCTION

The problems of dynamic buckling of structures are usually associated with their response to rapidly increasing in-surface compressive loads or time-dependent in-surface displacements of the boundaries. The problems of the first type are typical for structures subject to impact, while the second type is particularly important in the studies of the response of structures in testing machines. The analytical approach to the solution of the linear dynamic buckling problem in the case of axial load or displacement of the boundaries proportional to time was proposed by Hoff,[1] who investigated the response

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of a slightly curved elastic column with the ends axially displaced towards each other at a constant speed. The solution of the problem found in terms of Bessel functions appeared to be in good agreement with experimental results.[2] The problems of dynamic buckling of isotropic structures subject to loads proportional to time were treated numerically and experimentally by Vol'mir,[3,4] Ari-Gur *et al.*,[5] Babich and Khoroshun,[6] and Brigadirov and Tolokonnikov.[7] A survey of works related to dynamic buckling of cylindrical shells can be found in the recent book of Bogdanovich.[8]

The recent paper of Saigal *et al.*[9] presented the finite element analysis of dynamic buckling of thin shells and a bibliography on dynamic buckling of spherical shells subjected to step or impulsive load.

Ekstrom considered dynamic buckling of initially imperfect geometrically nonlinear simply-supported rectangular orthotropic plates subject to the compressive load increasing proportionally to time.[10] The problem, reduced to a single second-order nonhomogeneous differential equation with time-dependent coefficients, was solved numerically using a fourth-order Runge–Kutta method.

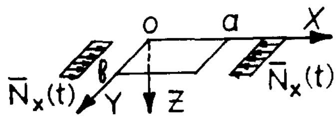
The theory of dynamic buckling of single degree of freedom systems due to step loading was developed by Budiansky and Hutchinson,[11,12] Budiansky[13] and Elishakoff.[14] They derived the relationships between the critical step load and the amplitude of the initial imperfection for structures with cubic, quadratic and quadratic-cubic nonlinearities as well as the conditions of imperfection-sensitivity or insensitivity.

In this paper dynamic buckling of simply-supported imperfect angle-ply plates due to a time-proportional axial load and a step load is considered.

In the first problem the nonlinear dynamic deformations of initially imperfect plates subjected to a time-dependent axial load are studied. The second problem is formulated as a particular case of the first one, i.e. initial imperfections are absent and the equations of motion are linear. This results in the question on the conditions of dynamic buckling of the perfect plate. Such a condition is found in the practical case where the plate is at rest at the instant of application of the load. The third problem deals with the dynamic response of an imperfect plate which was initially subject to an axial static load. The static deformations have to be determined in the first phase of the solution. Then the dynamic displacements due to the action of the time-dependent load are superimposed on the static solution.

2 DYNAMIC BUCKLING OF IMPERFECT PLATES: NONLINEAR PROBLEM

Consider an antisymmetrically laminated angle-ply rectangular plate subject to the axial compressive load of intensity $\bar{N}_x(t)$, which is a function of



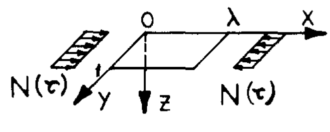


Fig. 1. Plate subjected to axial loading: (a) Dimensional scheme: $(x, y, z) = (X, Y, Z)/h$; (b) nondimensional scheme: $\lambda = a/b$.

time (Fig. 1). The sides of the plate along the X and Y axes have the lengths a and b respectively. The thickness of the plate is h. The rate of loading is such that the time required for the plate to obtain considerable deformation is much longer than the time required for the pressure wave to travel through the plate. This allows one to neglect the effect of the axial inertia.[15]

The equations of the nonlinear theory of laminated plates with initial imperfections, representing the generalization of the equations of the von Karman theory of isotropic plates, were published by Stavsky and Hoff[16] and Tennyson *et al.*[17] In this paper the nondimensional dynamic version of the equations used by Hui[18,19] is adopted:

$$L_d(w) + L_b(f) + \pi^4 w_{,\tau\tau} = f_{,yy}(w + w_0)_{,xx} + f_{,xx}(w + w_0)_{,yy} - 2f_{,xy}(w + w_0)_{,xy} \tag{1}$$

$$L_a(f) = L_b(w) + (w + 2w_0)_{,xy} w_{,xy} - (w + w_0)_{,xx} w_{,yy} - w_{0,yy} w_{,xx} \tag{2}$$

In eqns (1) and (2) w is the nondimensional out-of-plane displacement from the imperfect position, w_0 is the nondimensional initial imperfection and f is the nondimensional stress function:

$$w = \bar{w}/h; \quad w_0 = \bar{w}_0/h; \quad f = F/E_T h^3 \tag{3}$$

where $\bar{w}$, $\bar{w}_0$ and F are the out-of-plane displacement, initial imperfection and the Airy stress function, respectively, and E_T is the modulus of elasticity of a lamina in the transverse direction. The nondimensional time is defined as

$$\tau = \omega t \tag{4}$$

where

$$\omega = \pi^2 \sqrt{E_T h^3 / \rho b^4} \tag{5}$$

ρ being the mass of the plate per unit area.

The nondimensional coordinates are

$$x = X/b \qquad y = Y/b \tag{6}$$

so that the lengths of the plate sides are $\lambda = a/b$ and 1 (see Fig. 1). The linear operators in (1) and (2) are given by

$$\begin{aligned} L_a(\ ) &= a_{22}(\ )_{,xxxx} + (2a_{12} + a_{66})(\ )_{,xxyy} + a_{11}(\ )_{,yyyy} \\ L_b(\ ) &= (2b_{26} - b_{61})(\ )_{,xxxy} + (2b_{16} - b_{62})(\ )_{,xyyy} \\ L_d(\ ) &= d_{11}(\ )_{,xxxx} + 2(d_{12} + 2d_{66})(\ )_{,xxyy} + d_{22}(\ )_{,yyyy} \end{aligned} \tag{7}$$

where a_{ij}, b_{ij} and d_{ij} are the elements of the nondimensional matrices $[\bar{A}_{ij}]$, $[\bar{B}_{ij}]$ and $[\bar{D}_{ij}]$, defined by

$$\begin{aligned} [\bar{A}_{ij}] &= E_T h [A_{ij}]^{-1} \\ [\bar{B}_{ij}] &= -[A_{ij}]^{-1}[B_{ij}]/h \\ E_T h^3 [\bar{D}_{ij}] &= [D_{ij}] - [B_{ij}][A_{ij}]^{-1}[B_{ij}] \end{aligned} \tag{8}$$

The matrices of extensional, coupling and bending stiffnesses, $[A_{ij}]$, $[B_{ij}]$ and $[D_{ij}]$, are defined as usual in the theory of composite structures.

The initial imperfection and the transverse displacement of the simply-supported plate are given by

$$w_0 = W_0 \sin\frac{m\pi x}{\lambda} \sin n\pi y \qquad w = W(\tau) \sin\frac{m\pi x}{\lambda} \sin n\pi y \tag{9}$$

The substitution of (9) into (2) yields the nondimensional stress function:

$$f = c_0 W(\tau) \cos\frac{m\pi x}{\lambda} \cos n\pi y + [W^2(\tau) + 2W(\tau) W_0] \left(c_1 \cos\frac{2m\pi x}{\lambda} + c_2 \cos 2n\pi y \right) - \frac{N(\tau) y^2}{2} \tag{10}$$

where c_i are defined as in Ref. 18:

$$
\begin{aligned}
c_0 &= -C_b\left(\frac{m}{\lambda}, n\right) \Big/ C_a\left(\frac{m}{\lambda}, n\right) \\
c_1 &= n^2 \Big/ \left[32\left(\frac{m}{\lambda}\right)^2 a_{22}\right] \\
c_2 &= \left(\frac{m}{\lambda}\right)^2 \Big/ (32 n^2 a_{11})
\end{aligned}
\tag{11}
$$

Here

$$
\begin{aligned}
C_b\left(\frac{m}{\lambda}, n\right) &= (2b_{26} - b_{61})\left(\frac{m\pi}{\lambda}\right)^3 n\pi + (2b_{16} - b_{62})\frac{m\pi}{\lambda}(n\pi)^3 \\
C_a\left(\frac{m}{\lambda}, n\right) &= a_{22}\left(\frac{m\pi}{\lambda}\right)^4 + (2a_{12} + a_{66})\left(\frac{m\pi}{\lambda}\right)^2 (n\pi)^2 + a_{11}(n\pi)^4
\end{aligned}
\tag{12}
$$

The nondimensional load is

$$
N(\tau) = \frac{\bar{N}_x(\tau) b^2}{E_T h^3} \tag{13}
$$

The substitution of (9) and (10) into equation of motion (1) and the Galerkin procedure result in the following nonlinear differential equation:

$$
W(\tau)_{,\tau\tau} + k_1 W(\tau) - k_\tau W(\tau) + k_2 W^2(\tau) + k_3 W^3(\tau) = k_\tau W_0 \tag{14}
$$

The coefficients of this equation are

$$
\begin{aligned}
k_1 &= \frac{1}{\pi^4}\left[C_a\left(\frac{m}{\lambda}, n\right) + \frac{C_b^2(m/\lambda, n)}{C_a(m/\lambda, n)}\right] + 4\left(\frac{m}{\lambda}\right)^2 n^2 (c_1 + c_2) W_0^2 \\
k_\tau &= \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N(\tau) \\
k_2 &= 6\left(\frac{m}{\lambda}\right)^2 n^2 (c_1 + c_2) W_0 \\
k_3 &= 2\left(\frac{m}{\lambda}\right)^2 n^2 (c_1 + c_2)
\end{aligned}
\tag{15}
$$

where

$$C_d\left(\frac{m}{\lambda}, n\right) = \left[d_{11}\left(\frac{m}{\lambda}\right)^4 + 2(d_{12} + 2d_{66})\left(\frac{m}{\lambda}\right)^2 n^2 + d_{22}n^4\right]\pi^4 \quad (16)$$

Equation (14) can be integrated numerically for any analytical relationship $N(\tau)$ including the case of the load proportional to time as it was done in the numerical examples. The plate is usually at rest at the instant of application of the load, i.e. the initial conditions to be used are

$$W = W_{,\tau} = 0 \quad \text{at} \quad \tau = 0 \quad (17)$$

3 DYNAMIC BUCKLING OF PERFECT PLATES SUBJECT TO AXIAL LOAD PROPORTIONAL TO TIME: CONDITION OF BUCKLING

Consider a linear problem of dynamic buckling of the perfect plate subject to the load

$$N(\tau) = s\tau \quad (18)$$

where s is a coefficient representing the nondimensional rate of loading. The equation of motion is obtained from (14):

$$W(\tau)_{,\tau\tau} + (k_1 - c\tau)W(\tau) = 0 \quad (19)$$

with $c = (s/\pi^2)(m/\lambda)^2$.

The solution of such equations was considered by Hoff,[1] Kamke[20] and Watson.[21] Introducing the new variables

$$\xi = k_1 - c\tau \quad \eta = W(\tau) \quad (20)$$

one can transform (19) to

$$c^2(\mathrm{d}^2\eta/\mathrm{d}\xi^2) + \xi\eta = 0 \quad (21)$$

The solution of (20) is[20]

$$\eta = \xi^{0\cdot 5}[AJ_{1/3}(\xi^{1\cdot 5}/1\cdot 5c) + BY_{1/3}(\xi^{1\cdot 5}/1\cdot 5c)] \quad (22)$$

where A and B are constants of integration, and $J_{1/3}(\ldots)$ and $Y_{1/3}(\ldots)$

are the Bessel functions of the first and second kind respectively. Using the initial conditions (17) one obtains the set of two homogeneous algebraic equations:

$$AJ_{1/3}(\tilde{k}_1) + BY_{1/3}(\tilde{k}_1) = 0$$

$$A\left\{\frac{1}{2k_1^{0\cdot 5}}J_{1/3}(\tilde{k}_1) + \frac{k_1}{2c}[J_{-2/3}(\tilde{k}_1) - J_{4/3}(\tilde{k}_1)]\right\}$$
$$+ B\left\{\frac{1}{2k_1^{0\cdot 5}}Y_{1/3}(\tilde{k}_1) + \frac{k_1}{2c}[Y_{-2/3}(\tilde{k}_1) - Y_{4/3}(\tilde{k}_1)]\right\} = 0 \tag{23}$$

where

$$\tilde{k}_1 = k_1^{1\cdot 5}/1{\cdot}5c \tag{24}$$

The condition of dynamic buckling of perfect antisymmetrically laminated plates can be obtained from (23) if one requires the existence of a nonzero solution. This condition can be written as

$$Y_{1/3}(\tilde{k}_1)[J_{-2/3}(\tilde{k}_1) - J_{4/3}(\tilde{k}_1)] = J_{1/3}(\tilde{k}_1)[Y_{-2/3}(\tilde{k}_1) - Y_{4/3}(\tilde{k}_1)] \tag{25}$$

The coefficient $\tilde{k}_1$ can be evaluated from (25). This coefficient represents the relationship between k_1 and c corresponding to dynamic buckling. If the analysis indicates that elastic dynamic buckling does not occur but the load increases as given by (18) the plastic effects have to be considered.

4 DYNAMIC BUCKLING OF IMPERFECT PLATES SUBJECTED TO CONSTANT AXIAL LOAD PRIOR TO THE LOAD PROPORTIONAL TO TIME

Consider an imperfect plate subjected to axial loading $\bar{N}_{x0}$ = const. The plate will experience a static deformation w_s which can be found from the static version of eqns (1) and (2). The mode shape of the dynamic displacement is assumed to be the same as those of the static displacement and initial imperfection:

$$\{\tilde{w}, w_s, w_0\} = \{\tilde{W}(\tau), W_s, W_0\}\sin\frac{m\pi x}{\lambda}\sin n\pi y \tag{26}$$

The stress function is given by (10), where $N(\tau)$ must be replaced by

$$N_0 = \bar{N}_{x0}b^2/Eh^3 \tag{27}$$

and $W(\tau)$ must be replaced by W_s = const. The static out-of-plane displacement can be determined from

$$k_1 W_s - k_0 W_s + k_2 W_s^2 + k_3 W_s^3 = k_0 W_0 \tag{28}$$

where k_1, k_2 and k_3 are defined as in (15) and

$$k_0 = \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N_0 \tag{29}$$

Now the dynamic axial load $\bar{N}_{xt}(\tau)$ is applied to the plate. The motion of the plate due to this load will be superimposed on the basic static state. Equations of motion of the plate become

$$\begin{aligned} L_d(\tilde{w}) + L_b(\tilde{f}) + \pi^4 \tilde{w}_{,\tau\tau} = {} & \tilde{f}_{,yy}(w_s + w_0 + \tilde{w})_{,xx} + f_{,yy}\tilde{w}_{,xx} + \tilde{f}_{,xx}(w_s + w_0 + \tilde{w})_{,yy} \\ & + f_{,xx}\tilde{w}_{,yy} - 2\tilde{f}_{,xy}(w_s + w_0 + \tilde{w})_{,xy} - 2f_{,xy}\tilde{w}_{,xy} \end{aligned} \tag{30}$$

$$\begin{aligned} L_a(\tilde{f}) = {} & L_b(\tilde{w}) + (w_s + 2w_0 + \tilde{w})_{,xy}\tilde{w}_{,xy} + \tilde{w}_{,xy}w_{,xy} \\ & - (w_s + w_0 + \tilde{w})_{,xx}\tilde{w}_{,yy} - \tilde{w}_{,xx}w_{,yy} - w_{0,yy}\tilde{w}_{,xx} \end{aligned} \tag{31}$$

where $\tilde{w}$ and $\tilde{f}$ are dynamic fractions of the nondimensional out-of-plane displacement and the stress function defined by equations similar to (3).

The substitution of the expressions for the static and dynamic displacements and the initial imperfection into (31) yields

$$\begin{aligned} \tilde{f} = {} & c_0 \tilde{W} \cos\frac{m\pi x}{\lambda}\cos n\pi y + (2W_s + 2W_0 + \tilde{W})\tilde{W} \\ & \times \left(c_1 \cos\frac{2m\pi x}{\lambda} + c_2 \cos 2n\pi y\right) - \frac{N_x(\tau)y^2}{2} \end{aligned} \tag{32}$$

where

$$N_x(\tau) = \bar{N}_{xt}(\tau)b^2/E_T h^3 \tag{33}$$

Finally the equation of motion obtained from (30) after the substitution of w_0, w_s, $\tilde{w}$, f, $\tilde{f}$ and the application of the Galerkin procedure is

$$\tilde{W}(\tau)_{,\tau\tau} + p_1 \tilde{W}(\tau) - p_{\tau 2}(\tau)\tilde{W}(\tau) + p_2 \tilde{W}^2(\tau) + p_3 \tilde{W}^3(\tau) = p_{\tau 1}(\tau) \tag{34}$$

where

$$
\begin{aligned}
p_1 &= \frac{1}{\pi^4}\left[C_d\left(\frac{m}{\lambda}, n\right) + \frac{C_b^2(m/\lambda, n)}{C_a(m/\lambda, n)}\right] \\
&\quad + 2\left(\frac{m}{\lambda}\right)^2 n^2(c_1 + c_2)(3W_s^2 + 6W_0 W_s + 2W_0^2) - \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N_0 \\
p_2 &= 6\left(\frac{m}{\lambda}\right)^2 n^2(c_1 + c_2)(W_s + W_0) \\
p_3 &= 2\left(\frac{m}{\lambda}\right)^2 n^2(c_1 + c_2) \\
P_{\tau 1}(\tau) &= \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 (W_s + W_0)N_x(\tau) \\
p_{\tau 2}(\tau) &= \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N_x(\tau)
\end{aligned}
\tag{35}
$$

Equation (34) can be integrated numerically.

The initial conditions corresponding to the plate being at rest at the instant of load application are

$$\tilde{W} = \tilde{W}_{,\tau} = 0 \quad \text{at} \quad \tau = 0 \tag{36}$$

5 DYNAMIC BUCKLING OF PLATES SUBJECT TO STEP LOADING

If the plate is subject to the step load

$$
\begin{aligned}
N(\tau) &= N \quad \text{at} \quad \tau \geq 0 \\
N(\tau) &= 0 \quad \text{at} \quad \tau < 0
\end{aligned}
\tag{37}
$$

its dynamic response is governed by a modified eqn (14) where

$$k_\tau = \bar{k}_\tau = \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N = \text{constant} \tag{38}$$

The response can be found numerically by integration of the modified eqn (14) with the appropriate boundary conditions. Note that this equation is also the equation of free nonlinear vibrations of the plate statically compressed by the axial loads of intensity N. Therefore, the response can be expected to be a periodic motion; the period and the amplitude of this motion have to be determined.

The relationship between the nondimensional period and the amplitude of motion described by the modified eqn (14) can be also obtained analytically using a method similar to that applied by Elishakoff *et al.*[22] to investigate nonlinear free vibrations of imperfect bars.

The multiplication of both parts of the modified eqn (14) by $\mathrm{d}W$ and the integration yield

$$\begin{aligned} 1/2W^2(\tau)_{,\tau} = & -1/2(k_1-\bar{k}_\tau)W^2(\tau) - 1/3k_2W^3(\tau) - 1/4k_3W^4(\tau) \\ & + \bar{k}_\tau W_0 W(\tau) + C \end{aligned} \tag{39}$$

where C is a constant of integration. This constant can be found from the condition that the velocity of the plate is equal to zero at the time instant corresponding to the maximum deviation from the equilibrium:

$$W(\tau)_{,\tau} = 0 \quad \text{when} \quad W(\tau) = W_{\mathrm{m}} \tag{40}$$

Here W_{m} represents the amplitude of periodic motion.

Substituting (40) into (39) and integrating one obtains

$$\tau - \tau_0 = \int_{\bar{W}(\tau_0)}^{\bar{W}} \frac{\mathrm{d}\bar{W}}{[F(\bar{W})]^{1/2}} \tag{41}$$

where

$$\bar{W} = W(\tau)/W_{\mathrm{m}} \tag{42}$$

and

$$\begin{aligned} F(\bar{W}) = & (k_1-\bar{k}_\tau)(1-\bar{W}^2) + \tfrac{2}{3}k_2W_{\mathrm{m}}(1-\bar{W}^3) \\ & + \tfrac{1}{2}k_3W_{\mathrm{m}}^2(1-\bar{W}^4) - \bar{k}_\tau\bar{W}_0(1-\bar{W}) \end{aligned} \tag{43}$$

$$\bar{W}_0 = W_0/W_{\mathrm{m}} \tag{44}$$

As was shown by Jeffreys and Swirles[23] the period of motion described by (41) can be represented as

$$T = 2\int_{\bar{W}_2}^{\bar{W}_1} \frac{\mathrm{d}\bar{W}}{[F(\bar{W})]^{1/2}} \tag{45}$$

where $\bar{W}_1$ and $\bar{W}_2$ are two simple roots of the polynomial $F(\bar{W})$. One of these roots is $\bar{W} = 1$ as follows from (43). The second root should satisfy the equation

$$\bar{W}^3 + r\bar{W}^2 + s\bar{W} + s = 0 \tag{46}$$

where

$$r = 1 + \frac{4k_2}{3k_3 W_m}$$
$$s = r + 2\frac{k_1 - \bar{k}_T}{k_3 W_m^2} \tag{47}$$

The cubic equation (46) can be solved by standard methods; the discriminant of this equation being

$$D = (p/3)^3 + (q/2)^2 \tag{48}$$

In (48)

$$p = s - r^2/3$$
$$q = 2r^3/27 - rs/3 + s \tag{49}$$

The sufficient conditions for the discriminant (48) to be positive are

$$|W_0| << W_m \quad \text{and} \quad N << N_{cr} \tag{50}$$

where

$$N_{cr} = \pi^2 k_1 \left(\frac{\lambda}{m}\right)^2 \tag{51}$$

is the static buckling load of the plate corresponding to the mode with m and n half-waves in the direction of the axes x and y respectively.

If the conditions (50) are satisfied and the discriminant D is positive (which can be the case even if (50) is not satisfied) eqn (46) has one real and two complex roots. Then the period of nonlinear motion can be found exactly in terms of a complete elliptic integral following the procedure outlined in Ref. 22. The relationship $T(W_m)$ represents the solution of the problem of dynamic buckling of the nonlinear imperfect plate subject to step loading.

The condition of dynamic buckling of linear perfect plates subject to step loading is immediately obtained from the equation

$$W(\tau)_{,\tau\tau} + \left[k_1 - \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N\right]W = 0 \tag{52}$$

The natural frequency turns out to be zero thus indicating buckling of the plate at the static buckling load given by (51).

The solution of the problem of dynamic buckling of the plate subjected to constant axial load prior to the step loading is very similar to that outlined in Section 4. The modified eqn (34) has constant coefficients

$$\begin{aligned} p_{\tau 1}(\tau) &= \bar{p}_{\tau 1} = \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 (W_s + W_0)N \\ p_{\tau 2}(\tau) &= \bar{p}_{\tau 2} = \frac{1}{\pi^2}\left(\frac{m}{\lambda}\right)^2 N \end{aligned} \tag{53}$$

The integration of the modified eqn (34) is similar to that of the modified eqn (14).

Note that the relationships between the buckling step load and the initial imperfection as well as the condition of imperfection-sensitivity developed in Refs 11–14 are not applicable in the case considered here. This is due to the different structure of the governing equation (14) and the equation used by the authors of Refs 11–14. Equation (14) includes the terms $k_1\, W(\tau)$ and $k_2\, W^2(\tau)$ where the coefficients k_1 and k_2 depend on the initial imperfection. In the equation of motion of the model structures used in Refs 11–14 the corresponding coefficients are independent of the imperfection.

6 NUMERICAL EXAMPLES

Numerical integration of eqn (14) was carried out by the Runge–Kutta method. The material of the plate was graphite–epoxy with the following dimensionless characteristics:

$$E_L/E_T = 40, \quad G_{LT}/E_T = 0{\cdot}5, \quad \nu_{LT} = 0{\cdot}25$$

The lamination angle was ±30° and the number of layers was assumed to be large so that the bending stretching coupling was negligible.

The load was assumed to increase proportionally to time. The non-dimensional rate of loading was defined when the dimensional load

$$\bar{N}_x(t) = \bar{s}t \tag{54}$$

was replaced by $N(\tau)$ given by (18). Then

$$s = \frac{\bar{s}b^2}{\pi^2 E_T h^3} \sqrt{\frac{pb^4}{E_T h^3}} \tag{55}$$

The behavior of the square plate with different amplitudes of the initial imperfection is shown in Fig. 2. The displacements increase as a result of larger imperfections at the initial phase of motion. At the later phase the displacements exhibit a gradual growth combined with an oscillatory-type motion. This phase of motion is not shown since it corresponds to large deformations in which plastic effects are unavoidable. The effect of the rate of loading is illustrated in Fig. 3.

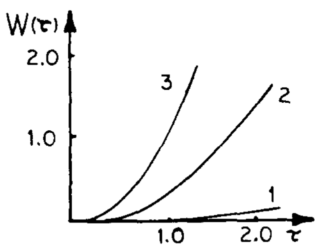


Fig. 2. Effect of initial imperfections on dynamic buckling of antisymmetrically laminated angle-ply plates. $\lambda = 1$, $m = n = 1$, $s = 3{\cdot}0$ (Curve 1, $W_0 = 0{\cdot}10$; Curve 2, $W_0 = 0{\cdot}25$; Curve 3, $W_0 = 0{\cdot}50$).

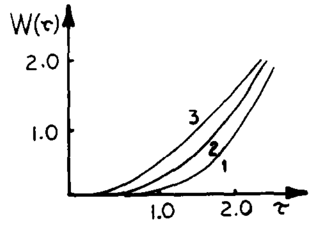


Fig. 3. Effect of rate of loading on dynamic buckling. $\lambda = 1, m = n = 1, W_0 = 0{\cdot}25$ (Curve 1, $s = 1{\cdot}0$; Curve 2, $s = 3{\cdot}0$; Curve 3, $s = 5{\cdot}0$).

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