How random is topological disorder?
Phase transitions and localization on random lattices

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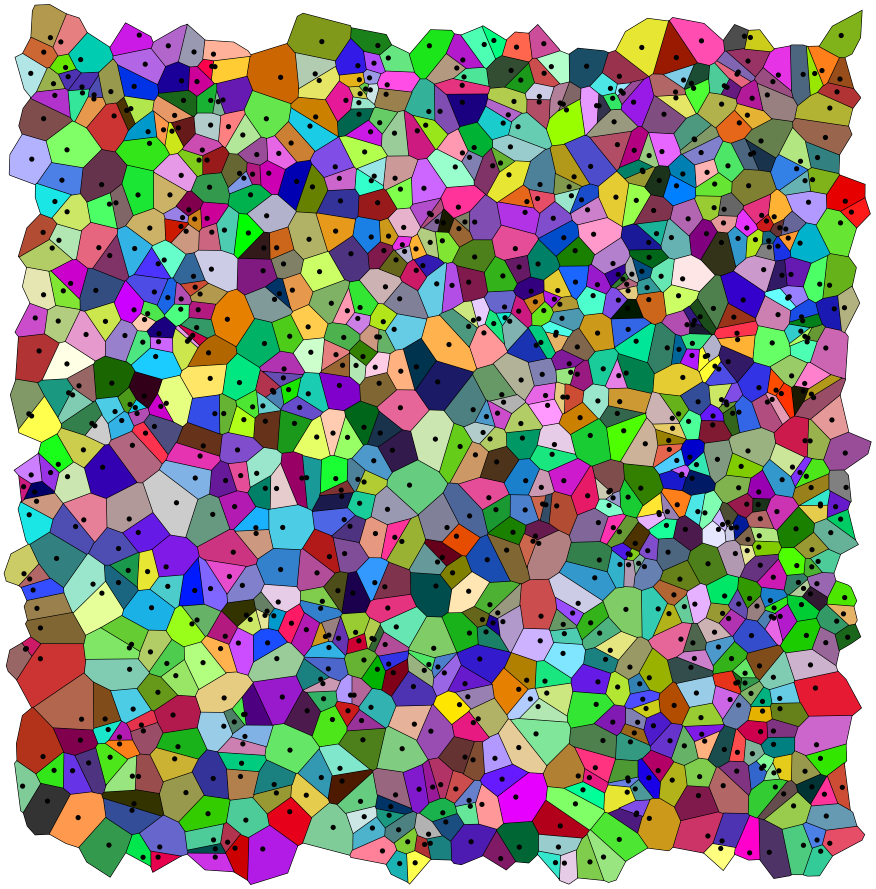

- Phase transitions, disorder, and universality
- Random Voronoi-Delaunay lattices
- Disorder fluctuations
- Generalization to broad class of lattices
- Phase transitions on random lattices
- Anderson localization on random lattices

Phys. Rev. Lett., **113**, 120602 (2014)
arXiv:

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Hatem Barghathi

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Martin Puschmann, Philipp Cain, and Michael Schreiber

Phase transitions, disorder, and universality

- effects of disorder on **thermal** and **quantum** phase transitions governed by universal criteria based on symmetry and dimensionality

Harris criterion governs stability of clean critical points: clean critical point stable against weak disorder if correlation length exponent fulfills $d\nu > 2$

Imry-Ma criterion controls phase coexistence in the presence of weak disorder: macroscopic phase coexistence stable against domain formation in dimensions $d > 2$ (discrete symmetry) or $d > 4$ (continuous symmetry)
$\Rightarrow$ 1st-order phase transitions rounded in dimensions $d \leq 2$.

Anderson localization governs spatial character of quantum states for weak disorder: all wave functions are localized in dimensions $d \leq 2$ (orthogonal case)

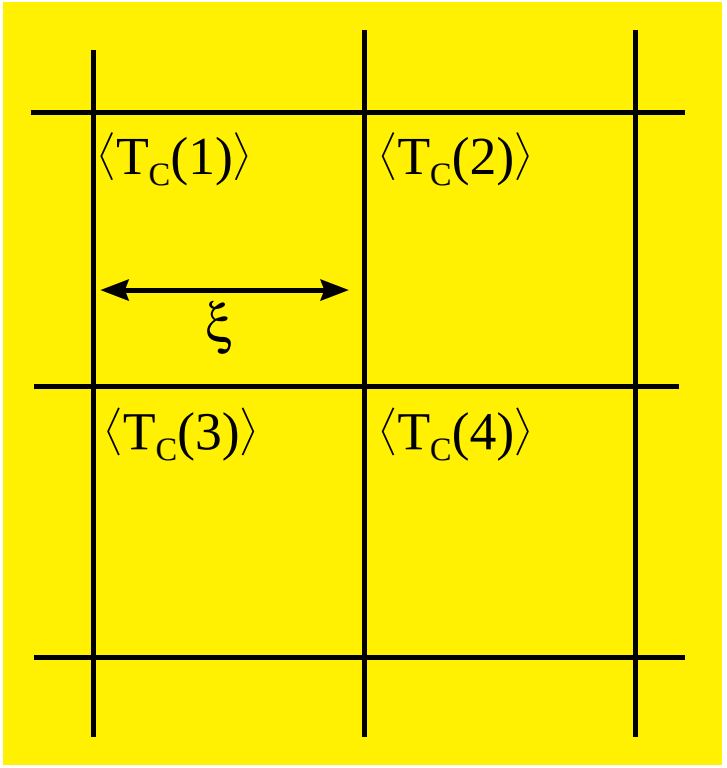
Harris criterion

Harris' insight:
variation of average local $T_c(x)$ in correlation volume must be smaller than distance from global T_c

variation of average T_c in volume ξ^d
$\Delta\langle T_c(x)\rangle \sim \xi^{-d/2}$

distance from global critical point
$T - T_c \sim \xi^{-1/\nu}$

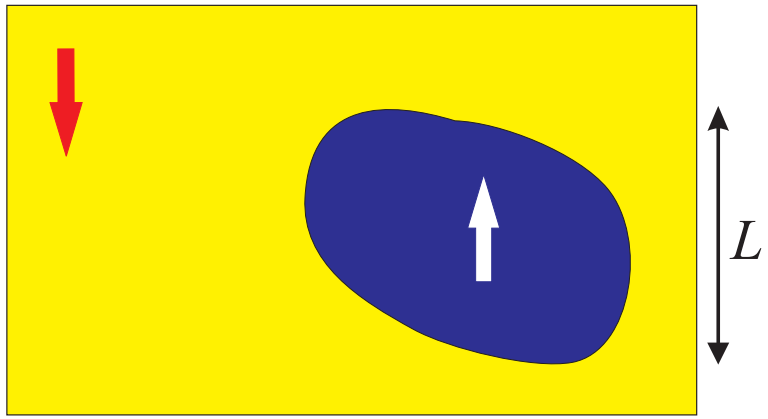
$\Delta\langle T_c(x)\rangle < T - T_c \qquad \Rightarrow \qquad d\nu > 2$



- if clean critical point fulfills Harris criterion $d\nu > 2 \Rightarrow$ stable against disorder
  example: 3D classical Heisenberg magnet: $\nu = 0.711$
- if $d\nu > 2$ is violated $\Rightarrow$ disorder relevant, character of transition changes
  example: 3D classical Ising magnet: $\nu = 0.628$

Imry-Ma criterion

Imry and Ma:
compare energy gain from forming domain that takes advantage of disorder with energy cost of domain wall



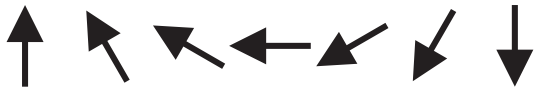
disorder energy gain:

$$E_{\rm dis} \sim L^{d/2}$$

domain wall energy:

$$E_{\rm DW} \sim \begin{cases} L^{d-1} & \text{(discrete)} \\ L^{d-2} & \text{(continuous)} \end{cases}$$

discrete



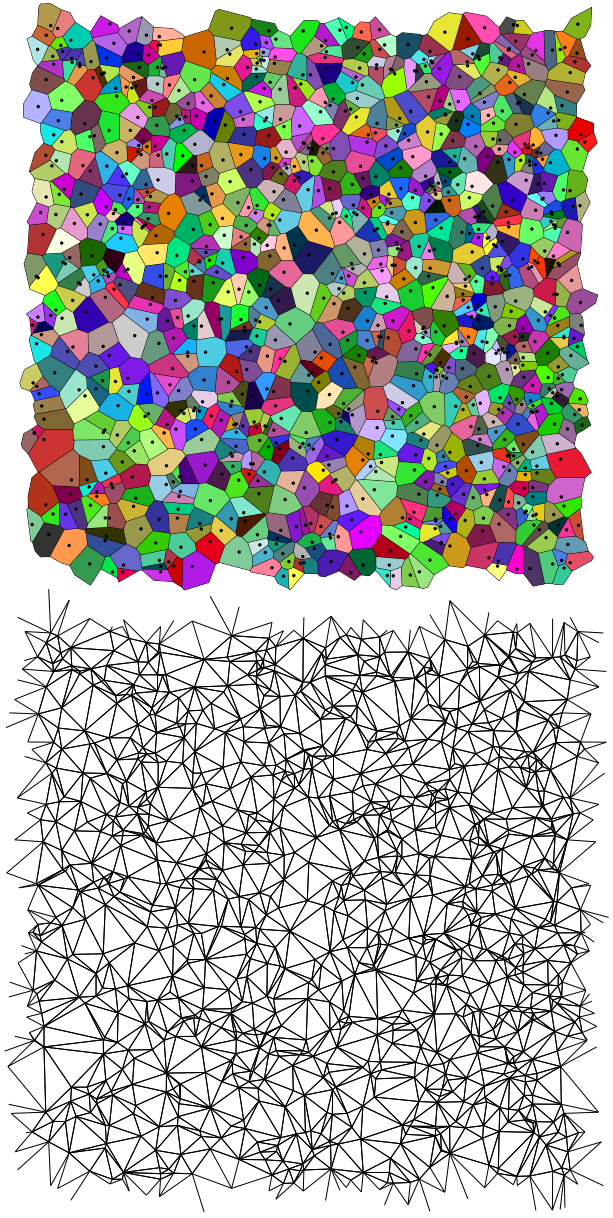
continuous

- if $E_{\rm dis} > E_{\rm RF}$ for $L \to \infty$: domain formation favorable
- phase coexistence **destroyed** by domain formation in $d \leq 2$ (discrete symmetry) or $d \leq 4$ (continuous symmetry)
- 1st-order phase transitions **rounded** in dimensions $d \leq 2$

Violations of universal criteria on random lattices

- criteria fulfilled for vast majority of systems with random interactions or impurities
- many apparent **violations** for **topological** disorder (random connectivity)

Random Voronoi lattice



3D Ising ferromagnet

• clean critical behavior even though Harris criterion violated, $d\nu < 2$

2D 8-state Potts model

• phase transition remains 1st order despite Imry-Ma criterion

2D Contact process

• clean critical behavior even though Harris criterion violated, no trace of exotic infinite-randomness physics found in diluted system

2D Tight-binding Hamiltonian

• energy-level statistics of metallic phase

What causes the failure of the Harris, Imry-Ma, and localization criteria on random (Voronoi) lattices??

- Phase transitions, disorder, and universality
- **Random Voronoi-Delaunay lattices**
- Disorder fluctuations
- Generalization to broad class of lattices
- Phase transitions on random lattices
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Voronoi-Delaunay construction

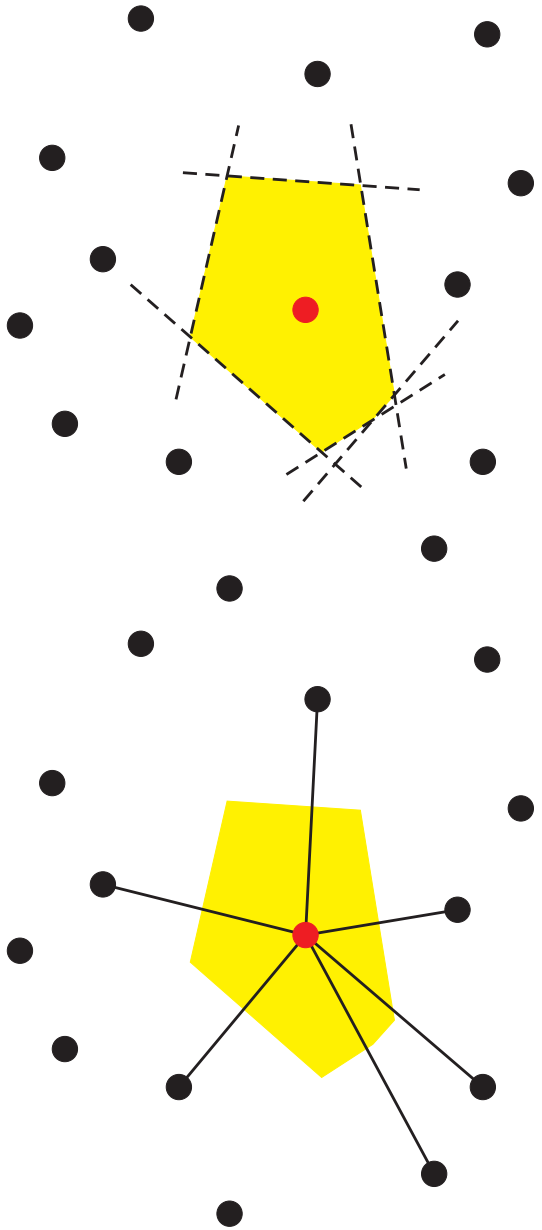
• construct cell structure from set of random lattice sites

Voronoi cell of site:

- contains all points in the plane (in space) closer to given site than to any other
- sites whose Voronoi cells share an edge (a face) considered neighbors

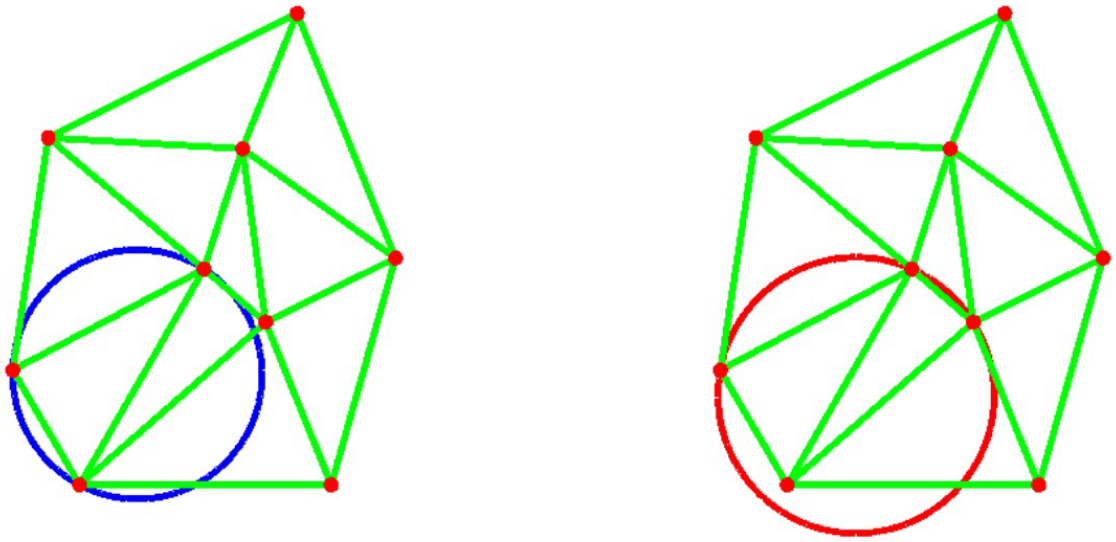
Delaunay triangulation (tetrahedrization):

- graph consisting of all bonds connecting pairs of neighbors
- dual lattice to Voronoi lattice



Algorithms

- generating Voronoi lattice or Delaunay triangulation is prototypical problem in **computational geometry**
- many different algorithms exist
- efficient algorithm inspired by Tanemura et al., uses **empty circumcircle property** up to 5000^2 sites in 2d and 400^3 sites in 3d
- computer time scales roughly **linearly** with number of sites
- 10^6 sites in 2d: about 30 seconds on PC
  10^6 sites in 3d: about 3 min

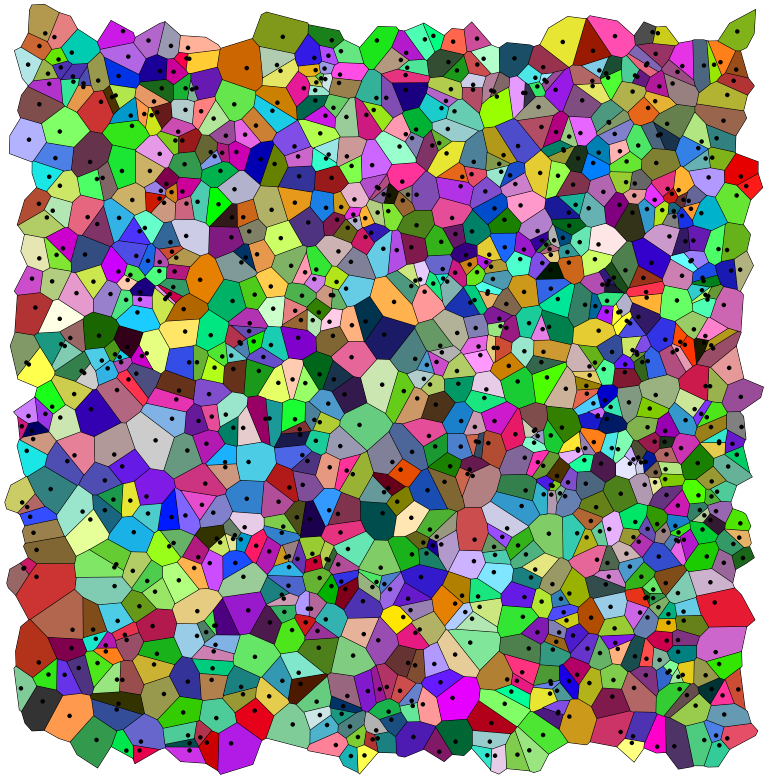


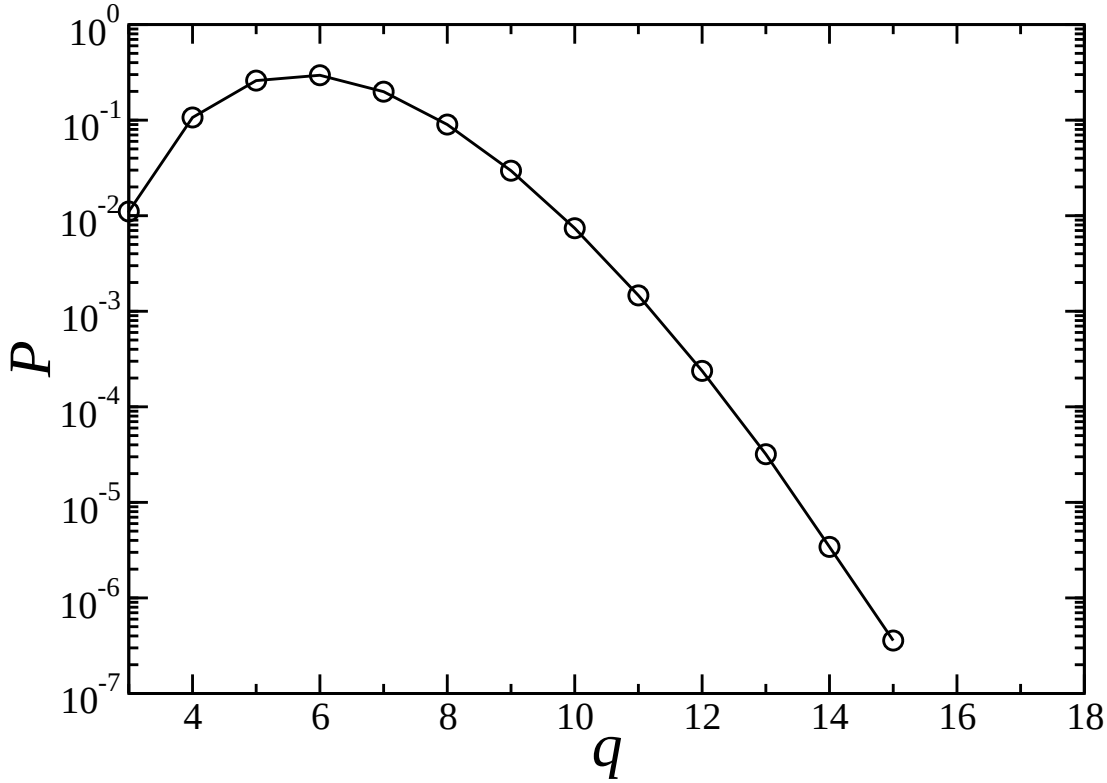
Properties of random Voronoi lattices

- lattice sites at independent random positions
- local coordination number q_i fluctuates
  2d: $\langle q \rangle = 6$, $\sigma_q \approx 1.33$
  3d: $\langle q \rangle = 2 + (48/35)\pi^2 \approx 15.54$, $\sigma_q \approx 3.36$

Moderate disorder!

- random connectivity (topology) generates disorder in physical system





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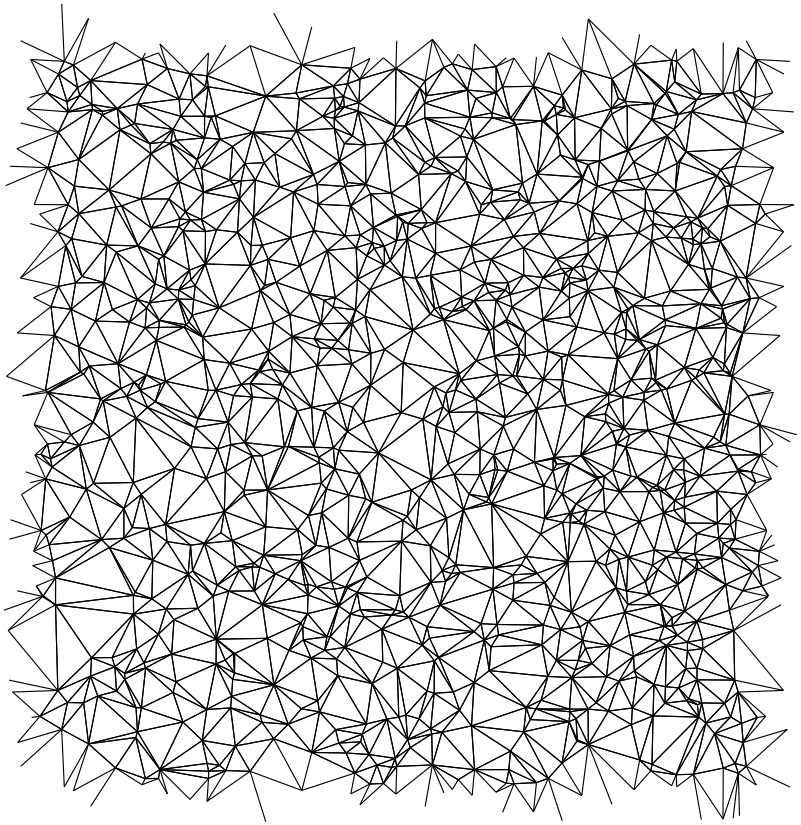
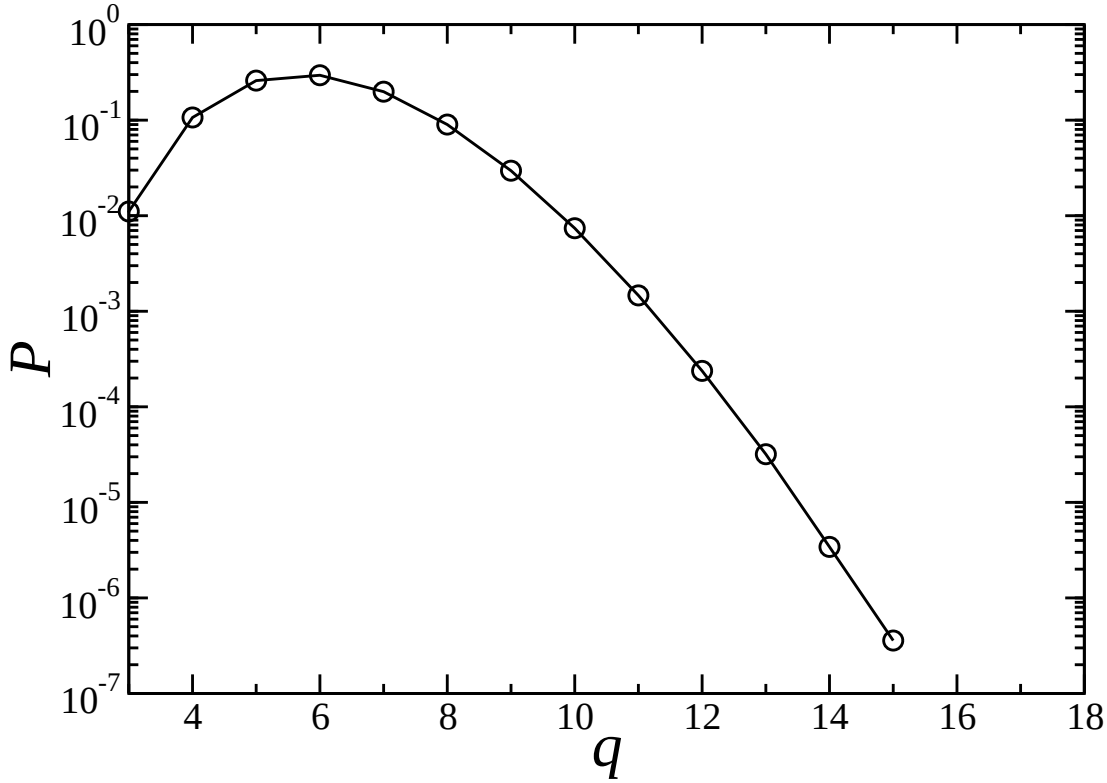
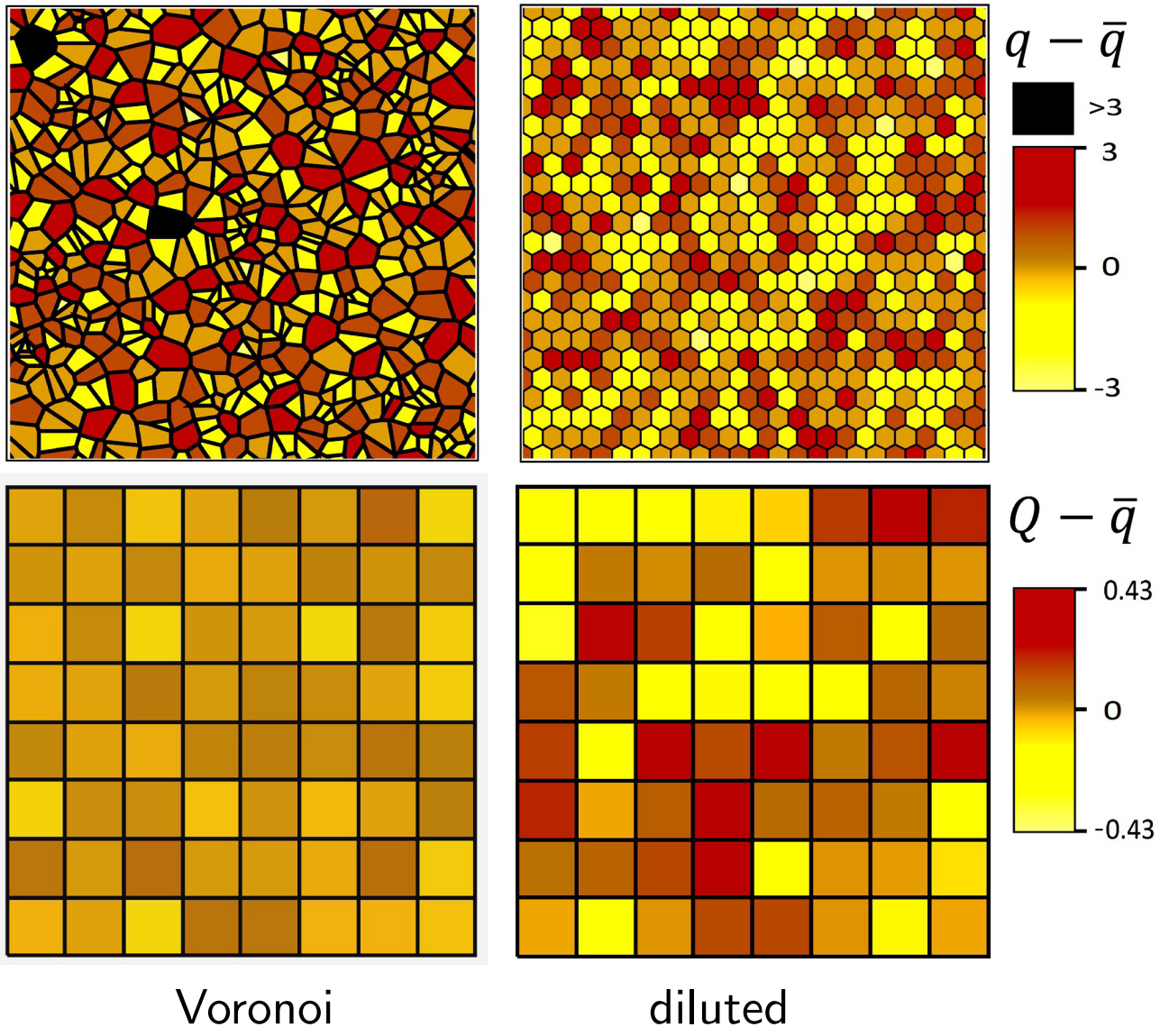




- Phase transitions, disorder, and universality
  - Random Voronoi-Delaunay lattices
    - **Disorder fluctuations**
  - Generalization to broad class of lattices
    - Phase transitions on random lattices
  - Anderson localization on random lattices

Coordination number fluctuations in 2d

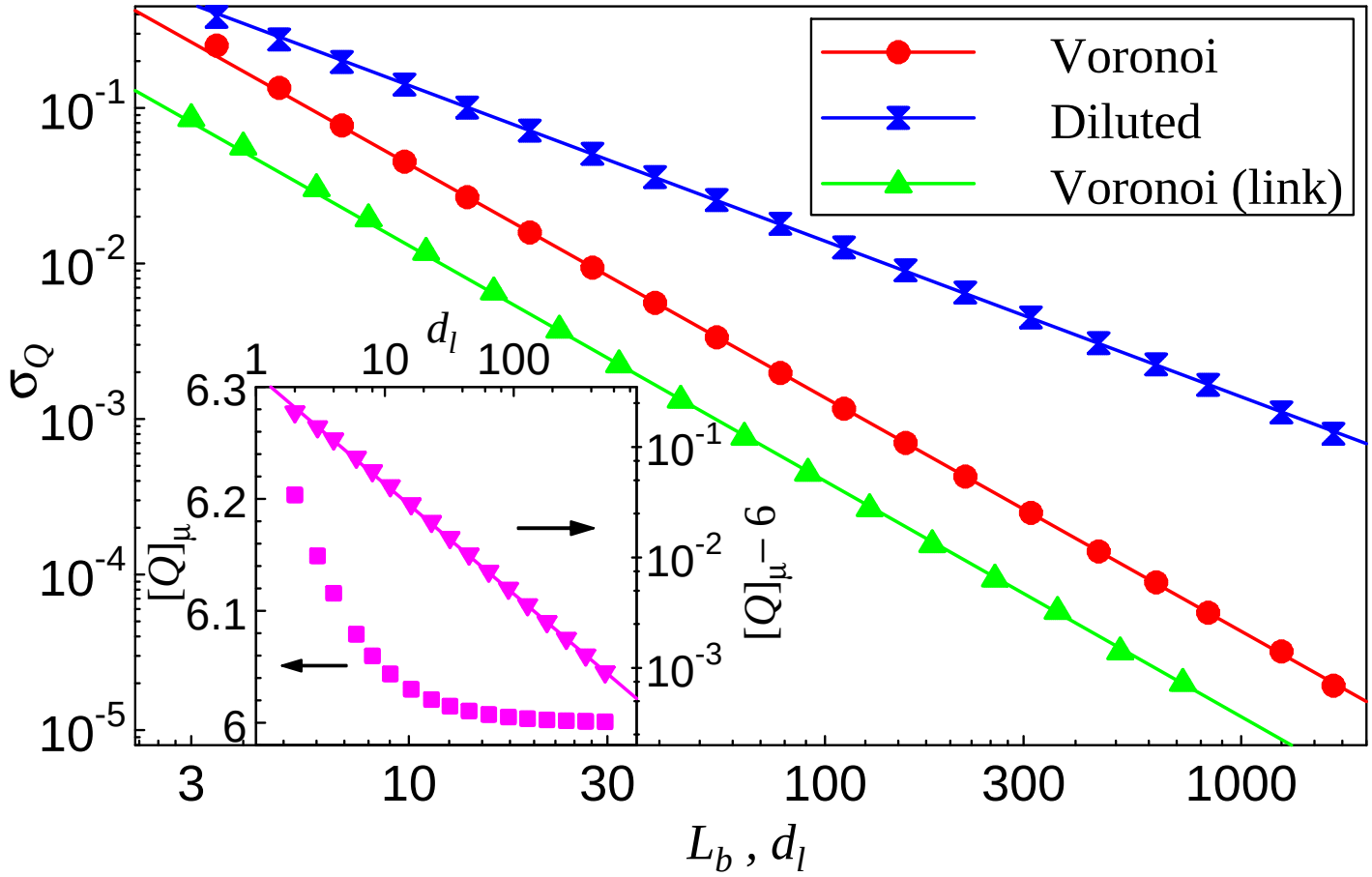


- divide large system into blocks of size L_b
- Calculate block-average coordination number

$$Q_\mu = \frac{1}{N_{b,\mu}} \sum_{i \in \mu} q_i$$

- fluctuations in Voronoi lattice suppressed

Coordination number fluctuations in 2d – II



- standard deviation
  $\sigma_Q^2(L_b) = \left[(Q_\mu - \bar{q})^2\right]_\mu$
- **Voronoi lattice:** $\sigma_Q \sim L_b^{-3/2}$
- **diluted lattice:** $\sigma_Q \sim L_b^{-1} \sim N_b^{-1/2}$
- also study **link-distance clusters**
- $\sigma_Q \sim L_b^{-3/2}$ as for the real-space clusters

Topological constraint

• What is the reason for the suppressed disorder fluctuations in the Voronoi lattice??

Euler equation for Delaunay triangulation:
(graph of N lattice sites, E edges, F facets, i.e., triangles)

$$N - E + F = \chi$$

χ: **Euler characteristic**, topological invariant of the underlying surface
torus topology (periodic boundary conditions): $\chi = 0$

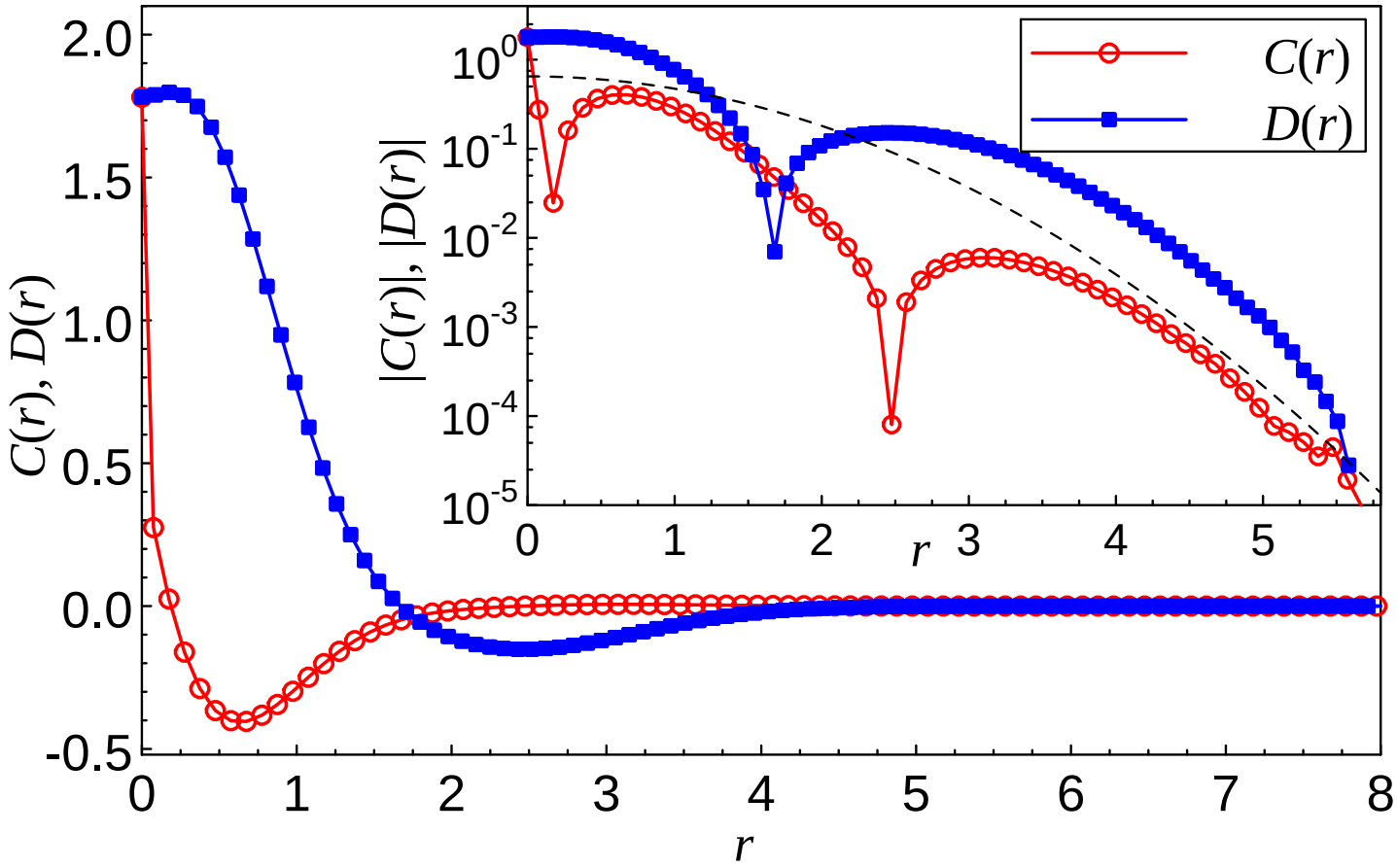
• each triangle has three edges, and each edge is shared by two triangles, $3F = 2E$
$\Rightarrow$ average coordination number precisely $\bar{q} = 6$ for any disorder configuration

• also follows from fixed angle sum of 180° in a triangle

Topological constraint introduces anticorrelations between disorder fluctuations

• fluctuations stem from surface: $\sigma_Q(L_b) \sim L_b^{(d-1)/2}/L_b^d = L_b^{-(d+1)/2} = L_b^{-3/2}$

Correlation function



$$C(\mathbf{r}) = \frac{1}{N}\sum_{ij}(q_i - \bar{q})(q_j - \bar{q})\delta(\mathbf{r} - \mathbf{r}_{ij})$$

$$\sigma^2_{Q,\mathrm{bulk}}(r) = D(r) = \frac{2\pi}{N_r}\int_0^r dr'\, r'\, C(r')$$

$\Rightarrow$ bulk contribution to fluctuations **negligible** beyond 5 or 6 n.n. distances

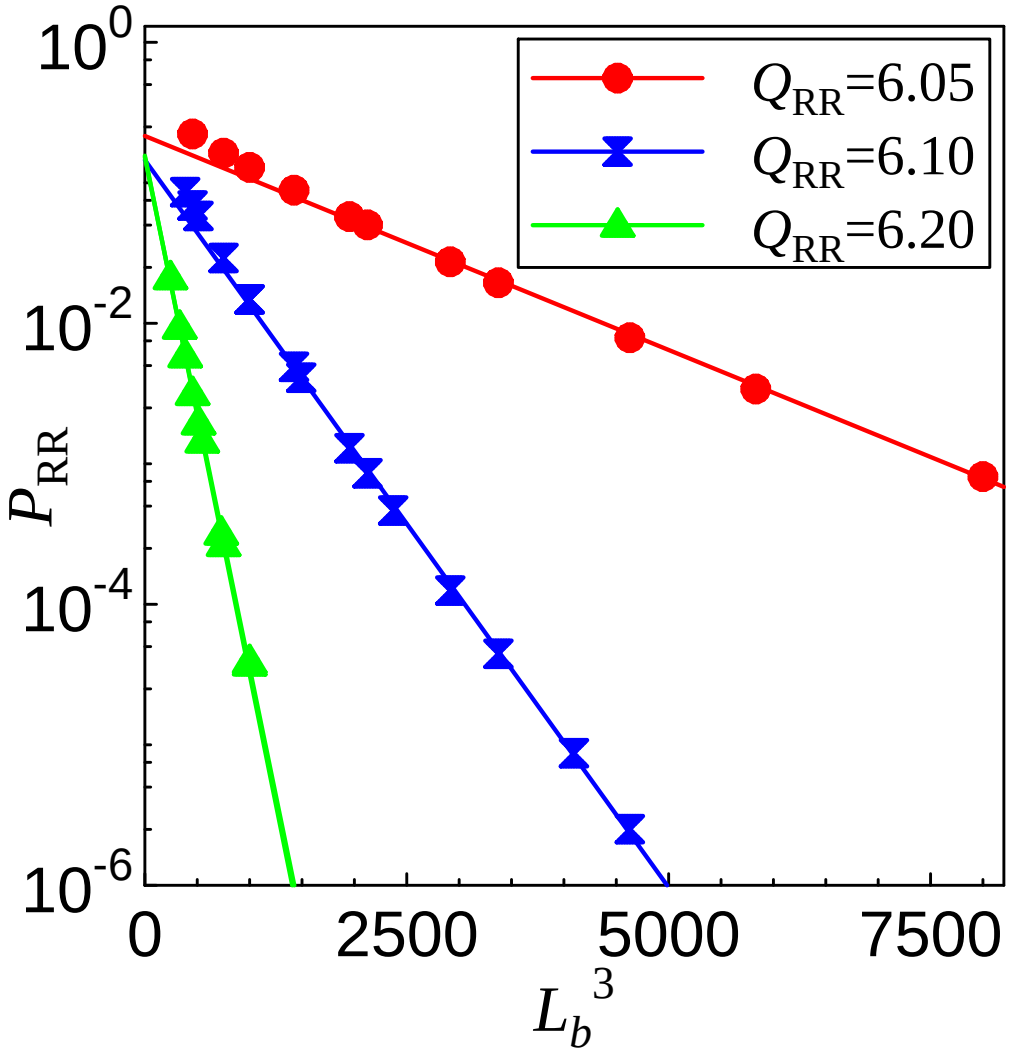
Rare regions

rare region:

- large block with average coordination number $Q_\mu > Q_{\rm RR}$ where $Q_{\rm RR}$ is constant larger than global average $\bar{q}$.
- **rare region probability**

$$P_{\rm RR} \sim \exp(-aL_b^3)$$

$\Rightarrow$ rare regions more strongly suppressed than in generic random systems for which $P_{\rm RR} \sim \exp(-aL_b^2)$



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- **Generalization to broad class of lattices**
  - Phase transitions on random lattices
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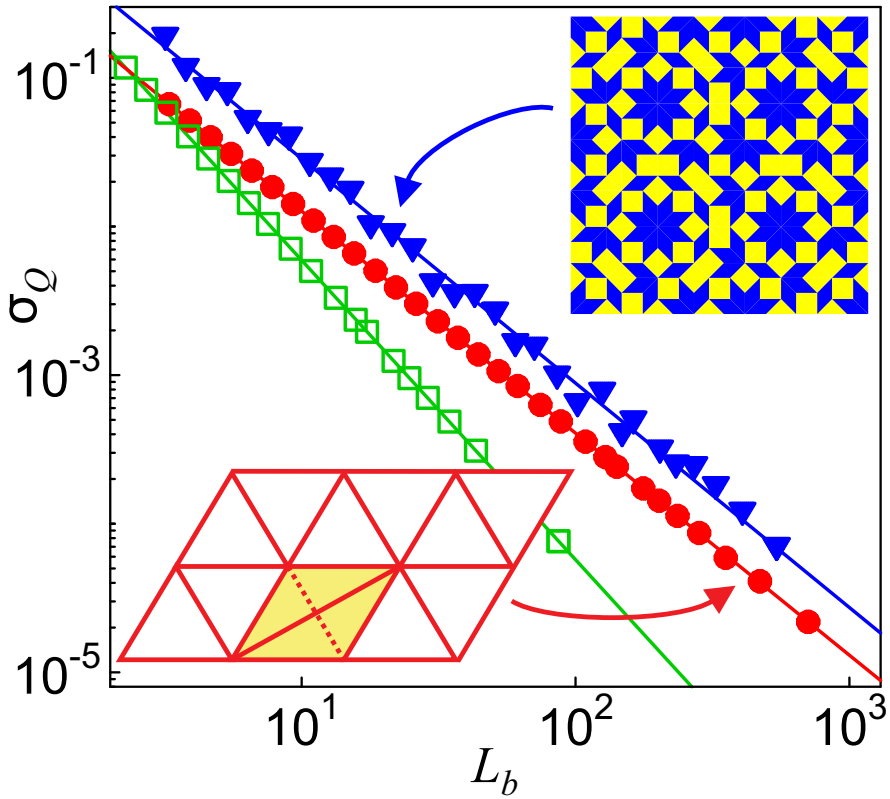
How general is the suppression of fluctuations?

Two dimensions:

- topological constraint: Euler eq. $N - E + F = \chi$ and triangle relation $3F = 2E$
- holds for all **random triangulations** (with short-range bonds)
- also holds for all **tilings with arbitrary quadrilaterals** (using $4F = 2E$)

$\Rightarrow$ **broad class** of random lattices with fixed total coordination, $\sigma_Q \sim L_b^{-3/2}$



Examples:

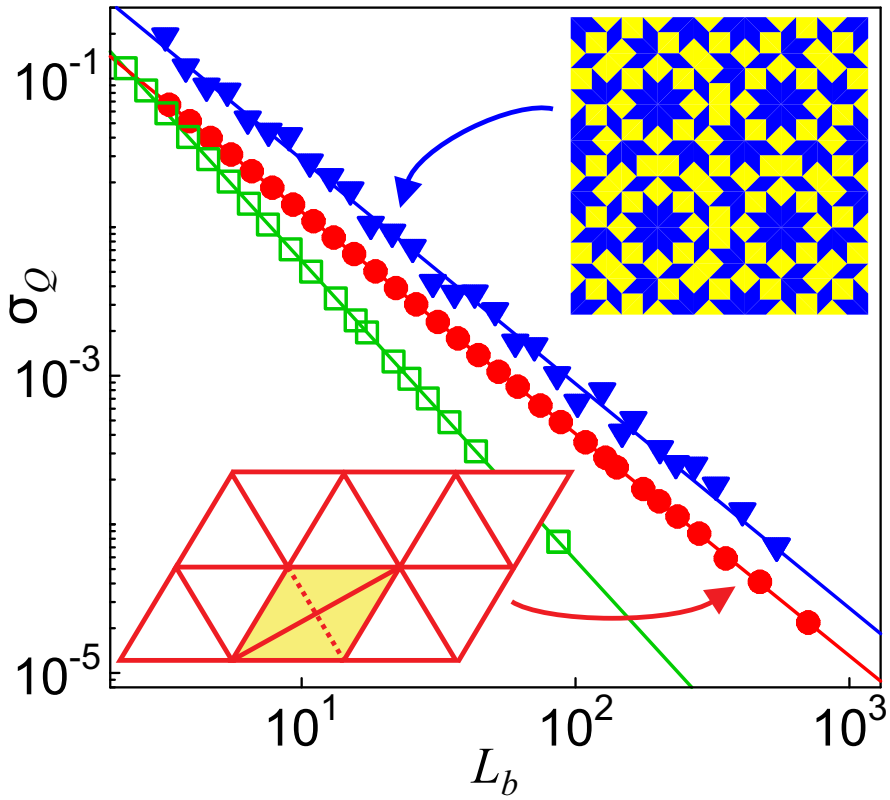
- random Voronoi lattices
- lattices with random bond-exchange defects
- quasiperiodic Penrose and Ammann-Beenker tilings

Three dimensions

Three-dimensional random Voronoi lattice

- Euler equation $N - E + F - C = \chi$ contains extra degree of freedom, number C of 3d cells (tetrahedra)
- total coordination number **not fixed** (solid angle sum in tetrahedron not constant)

$\Rightarrow$ 3d random Voronoi lattices **not in class** of lattices with suppressed fluctuations
(preliminary numerics: fluctuations still decay faster than central limit theorem)



3D lattices with fixed total coordination:

- random lattices built from rhombohedra (solid angle sum $\equiv 4\pi$)
- 3d versions of bond-exchange lattices
- coordination number fluctuations $\sigma_Q \sim L_b^{-2}$

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Physical systems on Voronoi lattices

Hamiltonians defined on random Voronoi lattice:

- various classical and quantum spin systems
- Josephson junction arrays and lattice bosons,
- nonequilibrium reaction-diffusion processes
- tight-binding Hamiltonian

How does the random connectivity change the character of the transition?

Disorder produced by coordination fluctuations

Example: classical magnet

- Hamiltonian $H = -J \sum_{\langle ij \rangle} S_i S_j$
  (interactions between nearest-neighbor sites of random lattice)
- local (mean) field $h_i = -J \sum_j' \langle S_j \rangle = -J q_i \langle S \rangle$

$\Rightarrow$ local coordination number determines local critical temperature

Coordination number fluctuations of random lattice determine (bare) disorder fluctuations of many-particle system.

- applies to all systems in which the total coupling strengths is a superposition of nearest neighbor contributions

Harris-Luck and Imry-Ma criteria

Stability of clean critical point:

- critical point stable if $\Delta\langle T_c(x)\rangle < T - T_c$
  $\Rightarrow \xi^{-(d+1)/2} < \xi^{-1/\nu} \quad \Rightarrow$ $(d+1)\nu > 2$ (rather than $d\nu > 2$)

- equivalent to Luck's generalization of Harris criterion, $d\nu > \dfrac{1}{1-\omega}$
  with wandering exponent $\omega = (d-1)/(2d)$

Phase coexistence:

- macroscopic phase coexistence possible if disorder energy gain less than domain wall energy, $E_{\rm dis} \sim L^{(d-1)/2}$ and $E_{\rm DW} \sim L^{d-1}$

$\Rightarrow$ phase coexistence possible, first-order phase transitions survive

Disorder anticorrelations of Voronoi lattice explain all apparent violations of the Harris and Imry-Ma criteria in 2d mentioned in the introduction

Disorder renormalizations

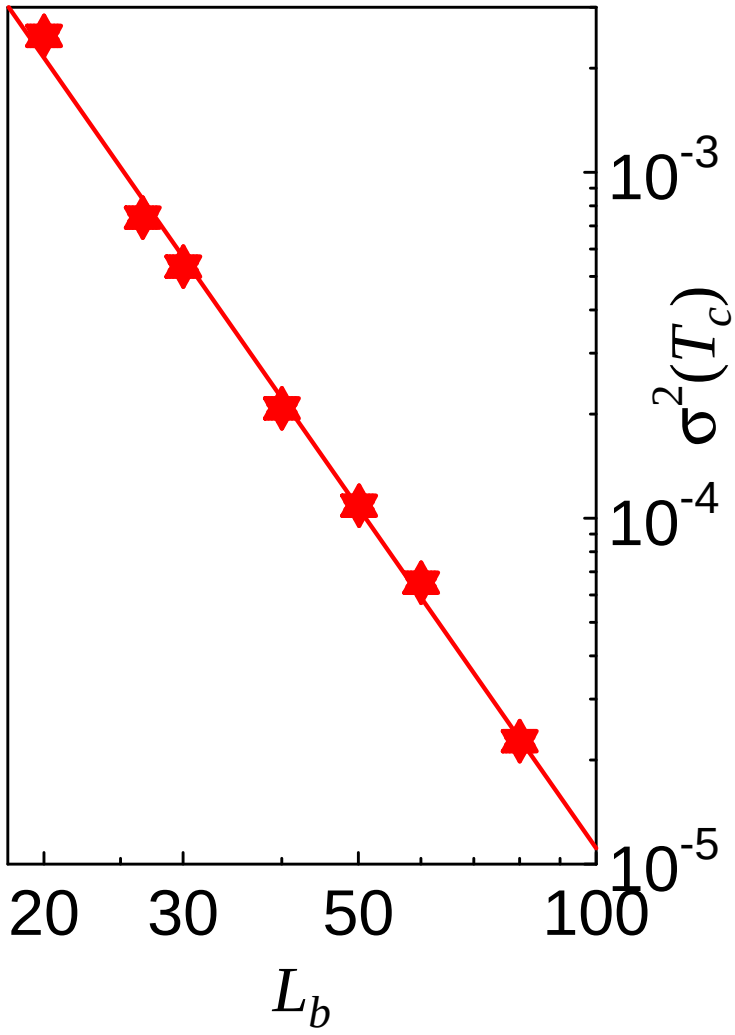
- coordination number determines **bare** disorder

What about **disorder renormalizations**?

- in some systems anticorrelations are protected, e.g., random transverse-field Ising chain (Hoyos et al., EPL, 93 (2011) 30004)
- in general, anticorrelations are **fragile**, weak uncorrelated disorder will be generated under RG

How strong is this uncorrelated disorder?

- fluctuations of local T_c of an Ising model on a random Voronoi lattice $\Rightarrow \sigma(T_c) \sim L_b^{-3/2}$
- uncorrelated disorder, if any, invisible for $L \lesssim 100$



$\Rightarrow$ uncorrelated disorder created by RG likely unobservable in experiment and numerics

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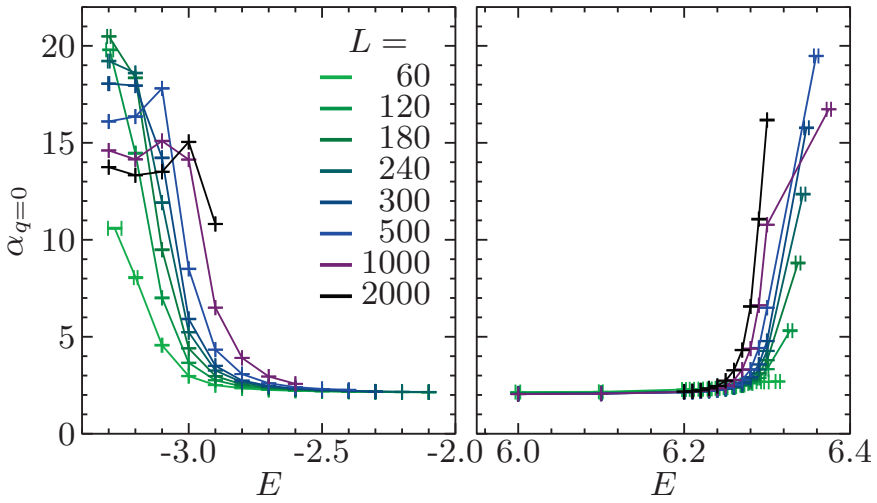
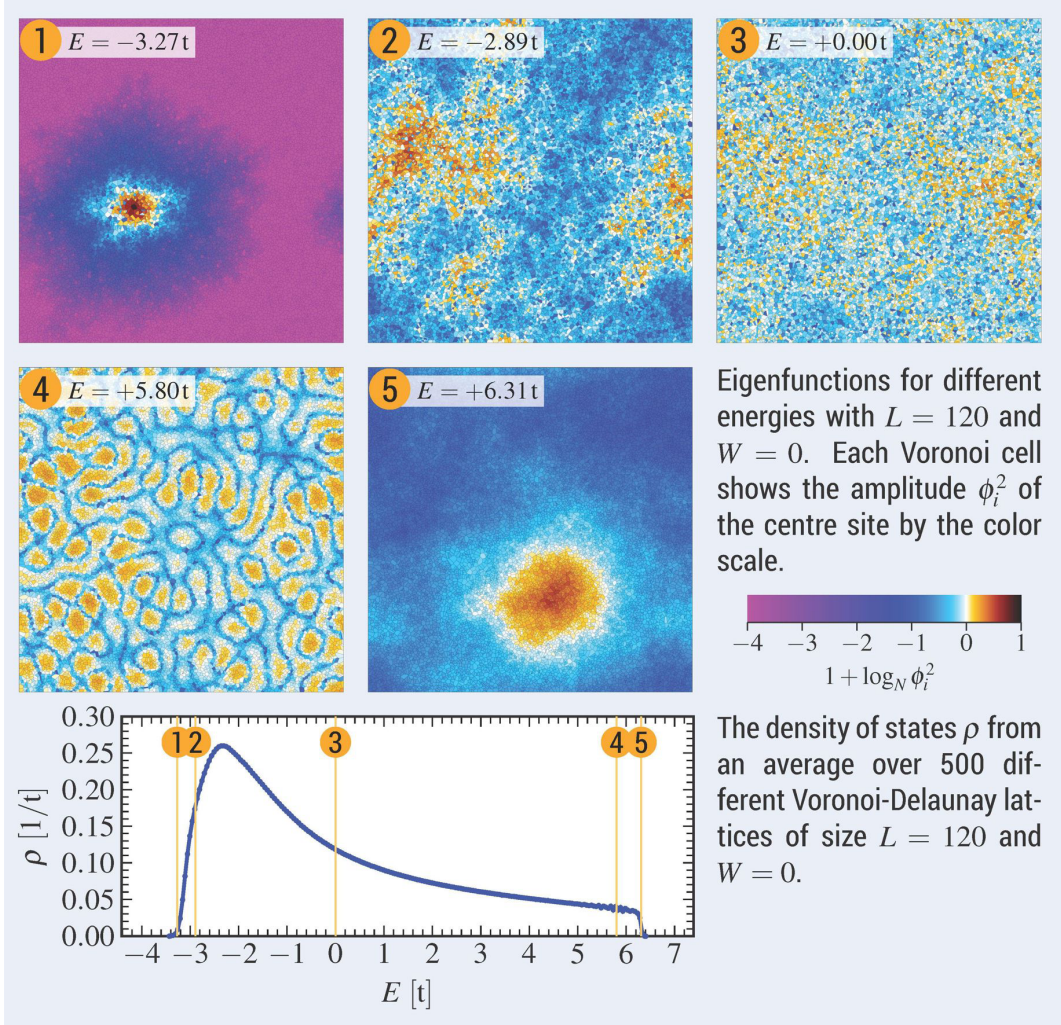
2D Anderson model on Voronoi-Delaunay lattice

$$H = \sum_{\langle i,j \rangle} |i\rangle \langle j| + \sum_i v_i |i\rangle \langle i|$$

$v_i \in [-W/2, W/2]$ random potentials

- multifractal analysis
- finite-size scaling





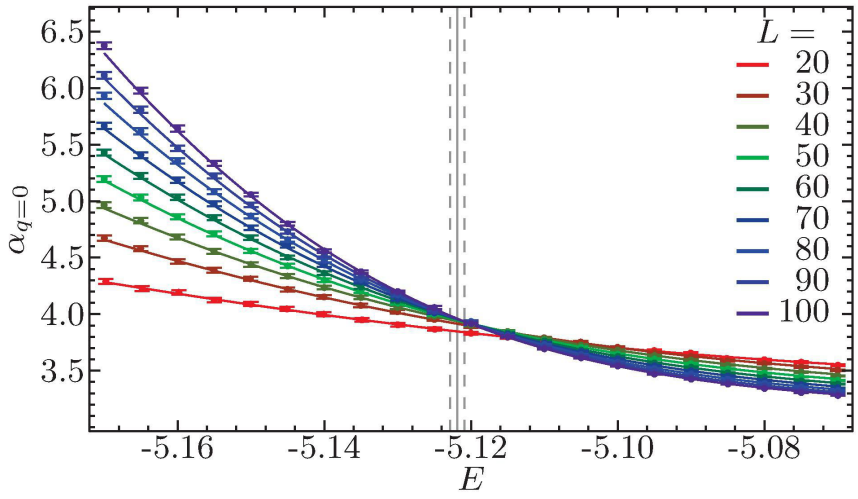
Eigenfunctions for different energies with $L = 120$ and $W = 0$. Each Voronoi cell shows the amplitude ϕ_i^2 of the centre site by the color scale.

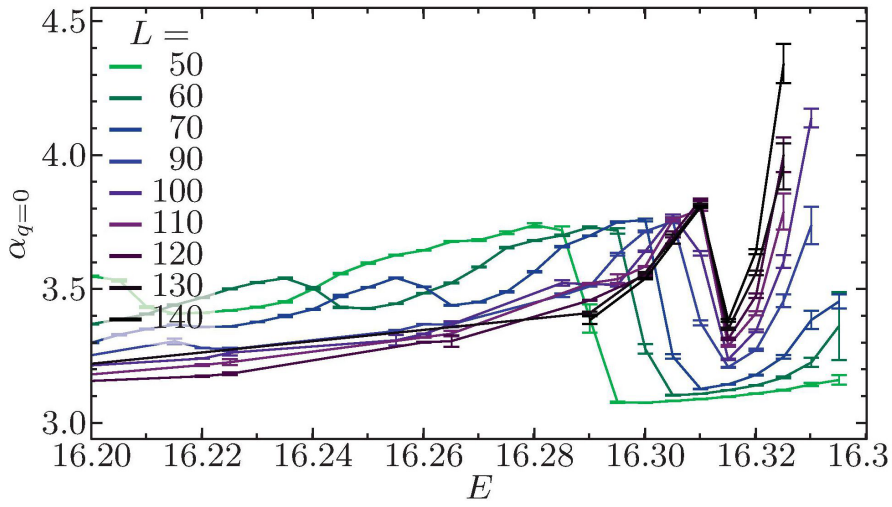
$1 + \log_N \phi_i^2$

The density of states ρ from an average over 500 different Voronoi-Delaunay lattices of size $L = 120$ and $W = 0$.

All states are localized, even in the absence of potential disorder, $W = 0$

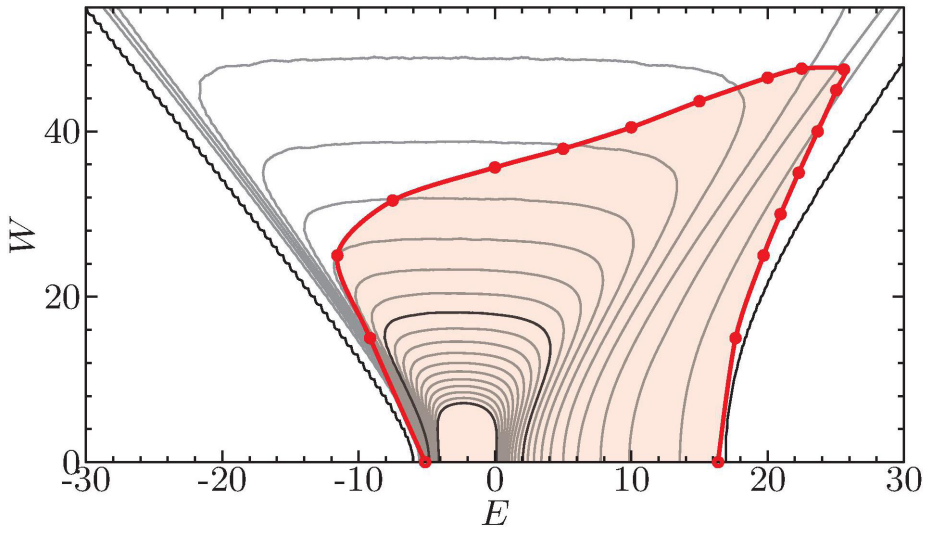
Localization on 3D Voronoi-Delaunay lattice





- two Anderson transitions close to the band edges, even for $W = 0$
- finite size scaling:
  correlation length exponent $\nu \approx 1.6$
- all results agree with usual **orthogonal universality class**

Phase diagram for $W \neq 0$



Conclusions

- Euler equation imposes **topological constraint** on coordination numbers in 2d

  $\Rightarrow$ broad class of random lattices with strong disorder **anticorrelations**
- in higher dimensions, lattices in this class exist, but they are not generic
- **modified Harris-Luck criterion** $(d+1)\nu > 2$
- **modified Imry-Ma criterion** allows first-order transitions to survive

This broad class of topological disorder is less random than generic disorder

Anderson localization:

- tight-binding model on random VD lattice is in usual **orthogonal universality class**
- universal properties of localization **not** influenced by anticorrelations $\Rightarrow$ quantum interference is crucial

H. Barghathi + T.V., Phys. Rev. Lett., **113**, 120602 (2014);

M. Puschmann, P. Cain, M. Schreiber + T.V., arXiv: