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A novel and accurate finite difference method for the fractional Laplacian and the fractional Poisson problem ☆

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A B S T R A C T

In this paper, we develop a novel finite difference method to discretize the fractional Laplacian $(-\Delta)^{\alpha/2}$ in hypersingular integral form. By introducing a splitting parameter, we formulate the fractional Laplacian as the weighted integral of a weak singular function, which is then approximated by the weighted trapezoidal rule. Compared to other existing methods, our method is more accurate and simpler to implement, and moreover it closely resembles the central difference scheme for the classical Laplace operator. We prove that for $u \in C^{3,\alpha/2}(\mathbb{R})$, our method has an accuracy of $\mathcal{O}(h^2)$ *uniformly for any* $\alpha \in (0,2)$, while for $u \in C^{1,\alpha/2}(\mathbb{R})$, the accuracy is $\mathcal{O}(h^{1-\alpha/2})$. The convergence behavior of our method is consistent with that of the central difference approximation of the classical Laplace operator. Additionally, we apply our method to solve the fractional Poisson equation and study the convergence of its numerical solutions. The extensive numerical examples that accompany our analysis verify our results, as well as give additional insights into the convergence behavior of our method.

1. Introduction

The fractional Laplacian $(-\Delta)^{\alpha/2}$ is a nonlocal generalization of the classical Laplacian that is often used to model diffusive processes. It arises in many areas of applications, including models for anomalous diffusion or dispersion [3,28], turbulent flows [25,3], porous media flows and pollutant transport [29], quantum mechanics [21,11], stochastic dynamics [15], and finance [7]. Contrary to the classical Laplacian, neither the theoretical properties of the fractional Laplacian nor its numerical treatments are yet fully understood. In particular, the nonlocality of the fractional Laplacian introduces considerable challenges for its numerical approximation. In this paper, we propose a novel and accurate finite difference method based on the weighted trapezoidal rule to discretize the fractional Laplacian.

Over $\mathbb{R}^n$, the fractional Laplacian is defined as the pseudo-differential operator with symbol $|\kappa|^\alpha$ [20,23]:

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) = \mathcal{F}^{-1}\big[|\kappa|^\alpha \mathcal{F}[u]\big], \qquad \text{for } \alpha > 0, \tag{1.1}$$

where $\mathcal{F}$ is the Fourier transform over $\mathbb{R}^n$ with associated inverse transform $\mathcal{F}^{-1}$. When $\alpha = 2$, expression (1.1) reduces to the well-known spectral representation of the standard Laplace operator $(-\Delta)$. The definition in (1.1) contextualizes the study of $(-\Delta)^{\alpha/2}$ conveniently within the field of Fourier analysis, which readily yields insights into some of its properties,

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such as regularity, and facilitates its approximation by means of efficient numerical methods based on fast Fourier transforms. It is, however, not immediately clear how (1.1) can be used to define the fractional Laplacian over non-periodic bounded domains. Multiple generalizations have been suggested in the literature, based on various equivalent definitions of $(-\Delta)^{\alpha/2}$ over $\mathbb{R}^n$ (see [20,23,17] and references therein). When restricted to bounded domains, these definitions give rise to different models, each with their own distinctive properties and their own paradigm for numerical approximation [24,10]. In this paper, we consider the definition of the fractional Laplacian in terms of the hypersingular integral [20,23]:

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) = c_{n,\alpha}\,\text{P.V.}\int_{\mathbb{R}^n} \frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}d\mathbf{y}, \qquad \text{for } \alpha \in (0,2), \tag{1.2}$$

where P.V. stands for the principal value integral, and $c_{n,\alpha}$ denotes a normalization constant

$$c_{n,\alpha} = \frac{2^{\alpha-1}\alpha\Gamma(\frac{\alpha+n}{2})}{\sqrt{\pi^n}\,\Gamma(1-\frac{\alpha}{2})}$$

with Γ representing the Gamma function. Probabilistically, the fractional Laplacian given by (1.2) represents the infinitesimal generator of a symmetric α-stable Lévy process [6,5].

Expression (1.2) clearly highlights the nonlocal nature of the operator $(-\Delta)^{\alpha/2}$, since the value of $(-\Delta)^{\alpha/2}u$ at any point $\mathbf{x} \in \mathbb{R}^n$ depends on the entire range of values $\{u(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\}$. In [9,8], the authors situate the fractional Laplacian within a larger class of nonlocal diffusion operators and consider variational formulations of nonlocal diffusion problems on bounded domains in $\mathbb{R}^n$. They show that volume constraints, that specify the solution over a non-degenerate region adjoining the domain, are the natural analogues of boundary conditions in the nonlocal case, and are necessary to ensure well-posedness.

The fractional Poisson equation with extended Dirichlet boundary conditions is the most well-known equation associated with the fractional Laplacian. It is defined on an open bounded domain $\Omega \in \mathbb{R}^n$ as [24,1,22]:

$$(-\Delta)^{\alpha/2}u(\mathbf{x}) = f(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{1.3}$$

$$u(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \Omega^c, \tag{1.4}$$

where $\Omega^c = \mathbb{R}^n\backslash\Omega$ represents the complement of Ω. The extended Dirichlet condition is readily incorporated in the definition (1.2) of the fractional Laplacian, by simply setting $u(\mathbf{y}) = 0$ in the integral when $\mathbf{y} \notin \Omega$. The regularity of the solution to (1.3)–(1.4) is studied in [22,26]; it shows that the solution $u \in C^{\alpha/2}(\mathbb{R}^n)$ (but not better), for $f \in L_\infty(\Omega)$ on a bounded Lipschitz domain Ω. More studies on its weak solutions can be found in [14,24]. Computationally, finite element methods are proposed in [1] to study the numerical solutions of (1.3)–(1.4).

So far, numerical methods for discretizing the fractional Laplacian (1.2) still remain scant, with the main numerical challenge stemming from the approximation of the hypersingular integral (1.2). To the best of our knowledge, there are two main finite difference methods in the literature [16,15]. In [15], a central difference method is combined with a "punched-hole" trapezoidal rule (i.e., discarding the influence of the region near the singularity), to approximate the fractional Laplacian, and the resulting method is applied to study stochastic differential equations with non-Gaussian α-stable Lévy motions. Recently, another finite difference method combined with interpolation was proposed in [16] to discretize the fractional Laplacian. It was proved that for $u \in C^4(\mathbb{R})$, this method has an accuracy of $\mathcal{O}(h^{2-\alpha})$ (resp. $\mathcal{O}(h^{3-\alpha})$), if piecewise linear (resp. quadratic) interpolation is used. Notably, the convergence rate depends on α and deteriorates as $\alpha \to 2^-$. Recall that for $u \in C^4(\mathbb{R})$, the central difference scheme for the classical Laplace operator $(-\Delta)$ has the convergence rate $\mathcal{O}(h^2)$. In this paper, we aim to develop a new finite difference scheme for the fractional Laplacian, such that as $\alpha \to 2$ it can recover the well-known central difference scheme for the classical Laplace operator.

By introducing a splitting parameter, the hypersingular integral of the fractional Laplacian can be formulated as the weighted integral of a weak singular function, which is then approximated by the composite weighted trapezoidal rule. This is the main novelty of our method, distinguishing it from other existing finite difference methods [16,15]. We prove that for $u \in C^{1,\alpha/2}(\mathbb{R})$, our method has an accuracy of $\mathcal{O}(h^{1-\alpha/2})$ in discretizing the fractional Laplacian. By choosing the splitting parameter optimally, we are in fact able to overcome the α-dependence of the accuracy estimate for $u \in C^{3,\alpha/2}(\mathbb{R})$ (a less stringent smoothness assumption than that used in [16]), achieving an accuracy of $\mathcal{O}(h^2)$ *for any* $\alpha \in (0,2)$. The main technique used in our numerical analysis is the weighted Montgomery identity. Our error estimates carry over to solutions of the fractional Poisson equation (1.3)–(1.4). Apart from a higher order of accuracy and a convergence rate independent of α, another merit of our method is that it is easy to implement computationally. Moreover, while generally giving rise to a full matrix, as is usually the case for the discretization of non-local operators, the discretized fractional Laplacian is a symmetric Toeplitz matrix, whose structure can be exploited through the use of fast algorithms [27,4,30].

We carry out extensive numerical experiments to study our method, both as a means of discretizing the fractional Laplacian and of approximating solutions to the fractional Poisson problem (1.3)–(1.4). In our numerical simulations, we confirm our theoretical error estimates, compare the performance of our approach with other existing methods, and investigate the influence of the splitting parameter. We also test the necessity of the underlying assumptions, by studying the method's convergence behavior for functions with different levels of smoothness. We observe that for less smooth functions, such as $u \in C^{1,\alpha/2}(\mathbb{R})$, the convergence rate in solving the fractional Poisson problem is better than that of discretizing the operator, consistent with the behavior of the central difference scheme for the classical Poisson problem.

The paper is organized as follows. In Section 2, we propose a finite difference method to discretize the fractional Laplacian, and its error estimates are presented in Section 3. In Section 4, we apply our numerical method to solve the fractional Poisson equation. Numerical examples are presented in Section 5 to examine the accuracy of our method in discretizing the fractional Laplacian and in solving the fractional Poisson equation. Some concluding remarks are made in Section 6.

2. Discretization of the fractional Laplacian

In this section, we outline our finite difference method for the fractional Laplacian $(-\Delta)^{\alpha/2}$. The novelty of our method is that we formulate the hypersingular integral representation of $(-\Delta)^{\alpha/2}$ as the *weighted integral* of a weaker singular function and then approximate it by the weighted trapezoidal rule. Here, one key idea is to introduce a *splitting parameter* that plays an important role in the accuracy of our method.

We will focus on the discretization of the fractional Laplacian on a one-dimensional bounded domain $\Omega = (-l, l)$ with extended homogeneous Dirichlet boundary conditions, i.e., $u(x) \equiv 0$ for $x \in \Omega^c$. Letting $\xi = |x - y|$, we can rewrite the one-dimensional fractional Laplacian in (1.2) as [11]:

$$(-\Delta)^{\alpha/2}u(x) = -c_{1,\alpha}\int_0^\infty \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\,d\xi. \tag{2.1}$$

Choose a constant $L = |\Omega| = 2l$. We can divide the integral in (2.1) into two parts:

$$(-\Delta)^{\alpha/2}u(x) = -c_{1,\alpha}\Bigg(\int_0^L \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\,d\xi + \int_L^\infty \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\,d\xi\Bigg). \tag{2.2}$$

Note that for any $x \in (-l, l)$ and $\xi \geq L$, there is $(x \pm \xi) \in \mathbb{R}\backslash(-l, l)$, and thus the function $u(x \pm \xi) \equiv 0$. Hence, the second integral in (2.2) reduces to

$$\int_L^\infty \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\,d\xi = -2u(x)\int_L^\infty \frac{1}{\xi^{1+\alpha}}d\xi = -\frac{2}{\alpha L^\alpha}u(x), \tag{2.3}$$

that is, it can be computed exactly.

Now, we will approximate the first integral in (2.2) numerically. Choose a positive integer K, and define the mesh size $h = 2l/K$. Denote grid points $\xi_k = kh$, for $0 \leq k \leq K$; evidently $\xi_K = L$. First, we introduce a splitting parameter $\gamma \in (\alpha, 2]$. The choice of γ is important in the accuracy of our method, and we will provide more discussion later. Then, we can formulate the first integral in (2.2) as:

$$\int_0^L \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^{1+\alpha}}\,d\xi = \int_0^L \phi_\gamma(x,\xi)\xi^{\gamma-(1+\alpha)}\,d\xi = \sum_{k=1}^K \int_{\xi_{k-1}}^{\xi_k} \phi_\gamma(x,\xi)\,\xi^{\gamma-(1+\alpha)}\,d\xi, \tag{2.4}$$

where for notational simplicity we define the function

$$\phi_\gamma(x,\xi) = \frac{u(x-\xi) - 2u(x) + u(x+\xi)}{\xi^\gamma}. \tag{2.5}$$

That is, the first integral in (2.2) can be viewed as a weighted integral of ϕ_γ, with $\xi^{\gamma-(1+\alpha)}$ representing the weight function. For $2 \leq k \leq K$, we use the weighted trapezoidal rule to approximate the integrals in (2.4), i.e.,

$$\int_{\xi_{k-1}}^{\xi_k} \phi_\gamma(x,\xi)\,\xi^{\gamma-(1+\alpha)}\,d\xi \approx \frac{1}{2}\Big(\phi_\gamma(x,\xi_{k-1}) + \phi_\gamma(x,\xi_k)\Big)\int_{\xi_{k-1}}^{\xi_k} \xi^{\gamma-(1+\alpha)}d\xi = \frac{1}{2(\gamma-\alpha)}\Big(\xi_k^{\gamma-\alpha} - \xi_{k-1}^{\gamma-\alpha}\Big)\Big(\phi_\gamma(x,\xi_{k-1}) + \phi_\gamma(x,\xi_k)\Big), \tag{2.6}$$

for any $\gamma = (\alpha, 2]$.

For $k=1$, we will divide our discussion into two cases based on different choices of the splitting parameter γ. If $\gamma=2$, the integral can be approximated by

$$\int_{\xi_0}^{\xi_1} \phi_\gamma(x,\xi)\xi^{\gamma-(1+\alpha)}d\xi = \int_{\xi_0}^{\xi_1} \phi_2(x,\xi)\xi^{1-\alpha}d\xi \approx \frac{1}{2-\alpha}\xi_1^{2-\alpha}\phi_2(x,\xi_1), \qquad \text{if } \gamma=2, \tag{2.7}$$

where the function $\phi_2(x,\xi_1)$ can be viewed as the central difference approximation to $u''(x)$. If the splitting parameter $\gamma \in (\alpha,2)$, we can approximate the integral for $k=1$ as:

$$\int_{\xi_0}^{\xi_1} \phi_\gamma(x,\xi)\,\xi^{\gamma-(1+\alpha)}d\xi \approx \frac{1}{2(\gamma-\alpha)}\xi_1^{\gamma-\alpha}\phi_\gamma(x,\xi_1), \qquad \text{if } \gamma \in (\alpha,2). \tag{2.8}$$

Assuming that u is smooth enough (e.g., $u \in C^2(\mathbb{R})$), (2.8) can be formally obtained using the weighted trapezoidal rule, i.e.,

$$\int_{\xi_0}^{\xi_1} \phi_\gamma(x,\xi)\,\xi^{\gamma-(1+\alpha)}d\xi \approx \frac{1}{2}\Big(\lim_{\xi\to 0}\phi_\gamma(x,\xi)+\phi_\gamma(x,\xi_1)\Big)\int_0^{\xi_1}\xi^{\gamma-(1+\alpha)}\,d\xi.$$

Since $\gamma \in (\alpha,\, 2)$, the above limit becomes zero, i.e.,

$$\lim_{\xi\to 0}\phi_\gamma(x,\xi) \approx \lim_{\xi\to 0}\xi^{2-\gamma}u''(x)=0.$$

As we will discuss later, (2.8) indeed provides a good approximation even for less smooth function, such as $u \in C^{1,\alpha/2}(\mathbb{R})$. In the special case of $\gamma = 1+\frac{\alpha}{2}$, the approximation in (2.8) coincides with (2.7). In fact, the scheme resulting from $\gamma = 1+\frac{\alpha}{2}$ or 2 leads to an optimal convergence rate; see detailed discussion in Section 3.

Combining (2.2)–(2.4) and (2.6)–(2.8), we obtain the numerical approximation of the fractional Laplacian $(-\Delta)^{\alpha/2}$ as:

$$(-\Delta)_{h,\gamma}^{\alpha/2}u(x) = -\frac{c_{1,\alpha}}{2(\gamma-\alpha)}\left[\kappa_\gamma\xi_1^{\gamma-\alpha}\phi_\gamma(x,\xi_1)+\sum_{k=2}^{K}\Big(\xi_k^{\gamma-\alpha}-\xi_{k-1}^{\gamma-\alpha}\Big)\Big(\phi_\gamma(x,\xi_{k-1})+\phi_\gamma(x,\xi_k)\Big)\right] + \frac{2c_{1,\alpha}}{\alpha L^\alpha}u(x), \qquad \text{for } x\in(-l,\,l), \tag{2.9}$$

where the constant $\kappa_\gamma=1$ for $\gamma\in(\alpha,2)$, while $\kappa_\gamma=2$ if $\gamma=2$. Define grid points $x_i=-l+ih$, for $0\le i\le K$. Using the definition of ϕ_γ in (2.5), we further obtain the fully discretized scheme as:

$$\begin{aligned}(-\Delta)_{h,\gamma}^{\alpha/2}u(x_i) = C_{\alpha,\gamma}^h\Bigg[&\left(\sum_{k=2}^{K-1}\frac{(k+1)^\nu-(k-1)^\nu}{k^\gamma}+\frac{K^\nu-(K-1)^\nu}{K^\gamma}+\big(2^\nu+\kappa_\gamma-1\big)+\frac{2\nu}{\alpha K^\alpha}\right)u(x_i)\\ &-\frac{2^\nu+\kappa_\gamma-1}{2}\Big(u(x_{i+1})+u(x_{i-1})\Big)\\ &-\frac{1}{2}\sum_{\substack{j=1,\\ j\ne i,\, i\pm1}}^{K-1}\frac{(|j-i|+1)^\nu-(|j-i|-1)^\nu}{|j-i|^\gamma}u(x_j)\Bigg], \qquad i=1,2,\ldots,K-1,\end{aligned} \tag{2.10}$$

where the coefficient $C_{\alpha,\gamma}^h = c_{1,\alpha}/(\nu h^\alpha)$, and we denote $\nu=\gamma-\alpha$ for notational simplicity.

Denote the vector $\mathbf{u}=(u(x_1),u(x_2),\cdots,u(x_{K-1}))^T$. Then, the scheme (2.10) can be expressed in matrix-vector form, i.e., $(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}=\mathrm{A}\mathbf{u}$, where A is the matrix representation of the fractional Laplacian, defined as

$$\mathrm{A}_{ij}=C_{\alpha,\gamma}^h\begin{cases}\displaystyle\sum_{k=2}^{K-1}\frac{\big(k+1\big)^\nu-\big(k-1\big)^\nu}{k^\gamma}+\frac{K^\nu-\big(K-1\big)^\nu}{K^\gamma}+\big(2^\nu+\kappa_\gamma-1\big)+\frac{2\nu}{\alpha K^\alpha}, & j=i,\\[2mm] \displaystyle-\frac{\big(|j-i|+1\big)^\nu-\big(|j-i|-1\big)^\nu}{2\,|j-i|^\gamma}, & j\ne i, i\pm1,\\[2mm] \displaystyle-\frac{1}{2}\big(2^\nu+\kappa_\gamma-1\big), & j=i\pm1,\end{cases}$$

for $i,j=1,2,\ldots,K-1$. It is easy to see that the matrix A is a symmetric Toeplitz matrix. The computation of $\mathrm{A}\mathbf{u}$ can be achieved efficiently by using the fast Fourier transform (FFT) [27,4,30], and the computational cost is $\mathcal{O}(K\ln K)$.

Remark 2.1. As $\alpha \to 2$, the scheme (2.10) with $\gamma = 1 + \frac{\alpha}{2}$ or 2 exactly reduces to the central difference scheme for the classical Laplace operator $-\Delta$, i.e.,

$$-\Delta_h u(x_i) = \frac{1}{h^2}\Big[-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})\Big], \qquad \text{for } i = 1, 2, \ldots, K-1,$$

independent of the mesh size h.

3. Error estimates

In this section, we provide error estimates on our method in discretizing the fractional Laplacian. The main technique used in our proof is an extension of the weighted Montgomery identity, which we will review in the following lemma:

Lemma 3.1. (Extension of weighted Montgomery identity [2,19]) *Let $f : [a, b] \to \mathbb{R}$ be a function such that the derivative $f^{(n)}$ exists for $n \geq 1$, and $w : [a, b] \to [0, \infty)$ be a weight function. For any $x \in [a, b]$, there is*

$$\int_a^b w(t) f(t) dt = \sum_{k=1}^{n} A_{k,w}(x) f^{(k-1)}(x) + (-1)^n \int_a^b P_{n,w}(x,t) f^{(n)}(t) dt, \tag{3.1}$$

where the weighted Peano kernel $P_{n,w}(x,t)$ is defined as

$$P_{n,w}(x,t) = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\int_a^t (t-s)^{n-1} w(s) ds, & \text{for } a \leq t \leq x, \\ \dfrac{1}{(n-1)!}\displaystyle\int_b^t (t-s)^{n-1} w(s) ds, & \text{for } x < t \leq b, \end{cases}$$

and the function

$$A_{k,w}(x) = \frac{(-1)^{k-1}}{(k-1)!}\int_a^b (x-t)^{k-1} w(t) dt, \qquad k = 1, 2, \ldots, n.$$

The identity (3.1) is an extension of the weighted Montgomery identity using Taylor's formula; see detailed proof in [2]. In the special case of $n = 1$ and $w(x) \equiv 1$, the identity (3.1) reduces to the classical Montgomery identity. From (3.1), we can obtain the following lemma:

Lemma 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a function such that the derivative $f^{(n)}$ exists for $n \geq 1$, and $w : [a, b] \to [0, \infty)$ be a weight function. Then, there is*

$$\begin{aligned} \int_a^b \left(f(t) - \frac{f(a)+f(b)}{2}\right) w(t) dt = {} & \frac{(-1)^n}{2}\int_a^b \Theta_{(a,b)}^{(n)}(t) f^{(n)}(t) dt \\ & + \frac{1}{2}\sum_{k=2}^{n} (-1)^{k-1}\Big(\Theta_{(a,b)}^{(n)}(a) f^{(k-1)}(a) + \Theta_{(a,b)}^{(n)}(b) f^{(k-1)}(b)\Big), \end{aligned} \tag{3.2}$$

where the function

$$\Theta_{(a,b)}^{(n)}(t) = \frac{1}{(n-1)!}\left(\int_b^t (t-s)^{n-1} w(s) ds + \int_a^t (t-s)^{n-1} w(s) ds\right), \qquad t \in [a, b].$$

Proof. Taking $x = a$ in (3.1) yields

$$\int_a^b f(t) w(t) dt = f(a)\int_a^b w(t) dt + \sum_{k=2}^{n} A_{k,w}(a) f^{(k-1)}(a) + (-1)^n \int_a^b P_{n,w}(a,t) f^{(n)}(t) dt. \tag{3.3}$$

Taking $x = b$ in (3.1), we obtain

$$\int_a^b f(t)w(t)dt = f(b)\int_a^b w(t)dt + \sum_{k=2}^{n} A_{k,w}(b) f^{(k-1)}(b) + (-1)^n \int_a^b P_{n,w}(b,t) f^{(n)}(t)dt. \tag{3.4}$$

Noticing the following properties of function $\Theta^{(n)}_{(a,b)}$:

$$\Theta^{(n)}_{(a,b)}(t) = P_{n,w}(a,t) + P_{n,w}(b,t),$$
$$\Theta^{(n)}_{(a,b)}(a) = P_{n,w}(a,a) = (-1)^{n-1} A_{n,w}(a),$$
$$\Theta^{(n)}_{(a,b)}(b) = P_{n,w}(b,b) = (-1)^{n-1} A_{n,w}(b),$$

we then obtain the identity (3.2) immediately by averaging (3.3) and (3.4). □

For simplicity, we will denote $\phi_\gamma(\xi) = \phi_\gamma(x,\xi)$. For an open set $\Omega \subset \mathbb{R}$ and $m \in (0,1]$, denote $C^{0,m}(\Omega)$ as the Hölder space of functions on Ω, i.e.,

$$C^{0,m}(\Omega) = \left\{ u \in C^0(\Omega) \,\middle|\, \sup_{\substack{x,\, y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^m} < \infty \right\}.$$

Furthermore, we denote

$$C^{n,m}(\Omega) = \left\{ u \in C^n(\Omega) \,\middle|\, u^{(k)} \in C^{0,m}(\Omega), \ \text{ for } k \in \mathbb{N} \text{ and } k \le n \right\},$$

for an integer $n \ge 0$. As the preparation, we will first study the properties of ϕ_γ in the following lemma:

Lemma 3.3. *Let $\alpha \in (0,2)$ and $\xi > 0$.*

(i). If $u \in C^{1,\alpha/2}(\mathbb{R})$, the derivative ϕ'_γ exists for any $\gamma \in (\alpha, 2]$. Furthermore, there are

$$\left|\phi_\gamma(\xi)\right| \le C_1 \xi^{\frac{\alpha}{2}+1-\gamma}, \qquad \left|\phi'_\gamma(\xi)\right| \le C_2 \xi^{\frac{\alpha}{2}-\gamma} \tag{3.5}$$

with C_1 and C_2 positive constants.

(ii). If $u \in C^{3,\alpha/2}(\mathbb{R})$, the derivative $\phi_2^{(n)}$ exists, for $n = 1$ or 2. Furthermore, there is

$$\left|\phi_2^{(n)}(\xi)\right| \le C \xi^{\frac{\alpha}{2}+1-n}, \qquad n = 1,\ 2 \tag{3.6}$$

with C a positive constant.

Lemma 3.3 can be proved by directly applying Taylor's theorem. Here, we will omit its proof for the sake of brevity. Define the norm

$$\|u\|_{\infty,\,\Omega} = \max_{x\in\Omega} |u(x)|.$$

Then, we have the following error estimates on our finite difference scheme (2.9) for $u \in C^{1,\alpha/2}(\mathbb{R})$:

Theorem 3.1. *Suppose that $u \in C^{1,\alpha/2}(\mathbb{R})$ has finite support on an open set $\Omega \subset \mathbb{R}$, and $(-\Delta)^{\alpha/2}_{h,\gamma}$ defined in (2.9) is a finite difference approximation of the fractional Laplacian $(-\Delta)^{\alpha/2}$. Then, for any $\gamma \in (\alpha, 2]$, there is*

$$\left\|(-\Delta)^{\alpha/2}u(x) - (-\Delta)^{\alpha/2}_{h,\gamma}u(x)\right\|_{\infty,\,\Omega} \le Ch^{1-\frac{\alpha}{2}}, \qquad \text{for } \alpha \in (0,2) \tag{3.7}$$

with C a positive constant depending on α and γ.

Proof. From (2.2)–(2.4) and (2.9), we obtain the error function

$$e^h_{\alpha,\gamma}(x) := (-\Delta)^{\alpha/2}u(x) - (-\Delta)^{\alpha/2}_{h,\gamma}u(x)$$
$$= -c_{1,\alpha}\Bigg[\Bigg(\int_{\xi_0}^{\xi_1} \phi_\gamma(\xi)\,\xi^{\gamma-(1+\alpha)}d\xi - \frac{\kappa_\gamma}{2}\phi_\gamma(\xi_1)\int_{\xi_0}^{\xi_1} \xi^{\gamma-(1+\alpha)}d\xi\Bigg)$$

$$+\sum_{k=2}^{K}\left(\int_{\xi_{k-1}}^{\xi_k}\phi_\gamma(\xi)\,\xi^{\gamma-(1+\alpha)}d\xi-\frac{\phi_\gamma(\xi_{k-1})+\phi_\gamma(\xi_k)}{2}\int_{\xi_{k-1}}^{\xi_k}\xi^{\gamma-(1+\alpha)}d\xi\right)\Bigg]$$
$$=-c_{1,\alpha}\big(I_\gamma+II_\gamma\big), \tag{3.8}$$

where the coefficient $\kappa_\gamma=1$ (resp. 2), if $\gamma\in(\alpha,2)$ (resp. $\gamma=2$).

For term I_γ, noticing $\xi_0=0$ and $\xi_1=h$ and using the triangle inequality, we obtain

$$|I_\gamma|=\left|\int_{\xi_0}^{\xi_1}\Big(\phi_\gamma(\xi)-\frac{\kappa_\gamma}{2}\phi_\gamma(\xi_1)\Big)\xi^{\gamma-(1+\alpha)}d\xi\right|$$
$$\le\int_0^h|\phi_\gamma(\xi)|\xi^{\gamma-(1+\alpha)}d\xi+\frac{\kappa_\gamma}{2}|\phi_\gamma(h)|\int_0^h\xi^{\gamma-(1+\alpha)}d\xi.$$

Using Lemma 3.3 (i) leads to

$$|I_\gamma|\le C\left(\int_0^h\xi^{\frac{\alpha}{2}+1-\gamma}\xi^{\gamma-(1+\alpha)}d\xi+h^{\frac{\alpha}{2}+1-\gamma}\int_0^h\xi^{\gamma-(1+\alpha)}d\xi\right)\le Ch^{1-\frac{\alpha}{2}}. \tag{3.9}$$

Now we move to the estimate of term II_γ. Using Lemma 3.2 with $n=1$ and weight function $w(\xi)=\xi^{\gamma-(1+\alpha)}$, we obtain

$$|II_\gamma|=\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\left(\phi_\gamma(\xi)-\frac{\phi_\gamma(\xi_{k-1})+\phi_\gamma(\xi_k)}{2}\right)\xi^{\gamma-(1+\alpha)}\,d\xi\right|$$
$$=\frac{1}{2}\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\Theta^{(1)}_{(\xi_{k-1},\,\xi_k)}(\xi)\,\phi_\gamma'(\xi)\,d\xi\right|.$$

From the definition of $\Theta^{(1)}_{(\xi_{k-1},\xi_k)}$ and the choice of weight function $w(\xi)$, we get

$$|II_\gamma|\le C\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\left(\int_{\xi_{k-1}}^{\xi}t^{\gamma-(1+\alpha)}dt+\int_{\xi_k}^{\xi}t^{\gamma-(1+\alpha)}dt\right)\big|\phi_\gamma'(\xi)\big|\,d\xi\right|$$
$$\le Ch\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\xi^{\gamma-(1+\alpha)}\big|\phi_\gamma'(\xi)\big|\,d\xi\right|$$
$$\le Ch\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\xi^{\gamma-(1+\alpha)}\xi^{\frac{\alpha}{2}-\gamma}\,d\xi\right|,$$

where the last inequality is obtained from Lemma 3.3 (i). Then, a direct calculation yields

$$|II_\gamma|\le Ch\big(h^{-\frac{\alpha}{2}}-L^{-\frac{\alpha}{2}}\big)\le Ch^{1-\frac{\alpha}{2}}. \tag{3.10}$$

Combining (3.8)–(3.10) yields

$$|e^h_{\alpha,\gamma}(x)|\le c_{1,\alpha}\big(|I_\gamma|+|II_\gamma|\big)\le Ch^{1-\frac{\alpha}{2}},\qquad \text{for } \gamma\in(\alpha,2).\quad\square$$

Remark 3.1. Theorem 3.1 shows that for $u\in C^{1,\alpha/2}(\mathbb{R})$, the accuracy of our method is $\mathcal{O}(h^{1-\alpha/2})$ for small mesh size h, i.e., its convergence for low regularity function is slow, especially as $\alpha\to2$. This result is consistent with the central difference scheme for the classical Laplace operator, which is not convergent if $u\in C^{1,1}(\mathbb{R})$ or even $C^2(\mathbb{R})$.

Although the convergence rate in this case is independent of the splitting parameter $\gamma\in(\alpha,2]$, our numerical studies (see Fig. 1) show that the numerical errors are considerably lower when choosing $\gamma=1+\frac{\alpha}{2}$ or $\gamma=2$.

Next, we will show that for a smooth enough function u, the accuracy of our method can be improved to $\mathcal{O}(h^2)$ uniformly for any $\alpha\in(0,2)$.

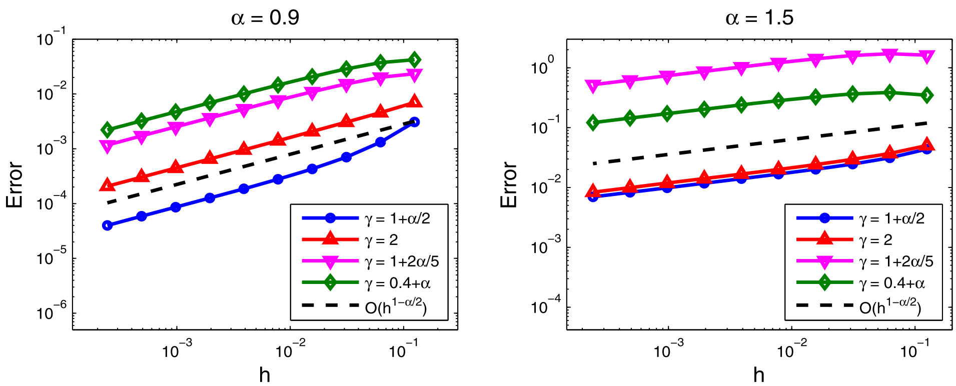

Fig. 1. Numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)^{\alpha/2}_{h,\gamma}u\|_{\infty,\Omega}$ for different choices of γ, where $u \in C^{1,\alpha/2}(\mathbb{R})$ is taken as in (5.2) with $s = 1$.

Theorem 3.2. *Suppose that $u \in C^{3,\alpha/2}(\mathbb{R})$ has finite support on an open set $\Omega \subset \mathbb{R}$, and $(-\Delta)^{\alpha/2}_{h,\gamma}$ defined in (2.9) is a finite difference approximation of the fractional Laplacian $(-\Delta)^{\alpha/2}$. If the parameter is chosen as $\gamma = 2$ or $\gamma = 1 + \frac{\alpha}{2}$, there is*

$$\left\|(-\Delta)^{\alpha/2}u(x) - (-\Delta)^{\alpha/2}_{h,\gamma}u(x)\right\|_{\infty,\,\Omega} \le Ch^2, \qquad \textit{for } \alpha \in (0,2) \tag{3.11}$$

with C a positive constant depending on α and γ.

Proof. First, we will focus on our proof for $\gamma = 2$. Starting with the error function in (3.8) with $\gamma = 2$, and using Lemma 3.2 with $n = 2$, we obtain

$$\begin{aligned}
e^h_{\alpha,2}(x) &= -c_{1,\alpha}\Bigg[\int_{\xi_0}^{\xi_1}\Big(\phi_2(\xi) - \phi_2(\xi_1)\Big)\xi^{1-\alpha}d\xi + \frac{1}{2}\sum_{k=2}^{K}\Bigg(\int_{\xi_{k-1}}^{\xi_k}\phi_2''(\xi)\,\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi)d\xi \\
&\qquad - \phi_2'(\xi_k)\,\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi_k) - \phi_2'(\xi_{k-1})\,\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi_{k-1})\Bigg)\Bigg] \\
&= -c_{1,\alpha}\Bigg[\int_{\xi_0}^{\xi_1}\Big(\phi_2(\xi) - \phi_2(\xi_1)\Big)\xi^{1-\alpha}d\xi + \frac{1}{2}\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\phi_2''(\xi)\,\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi)d\xi \\
&\qquad - \frac{1}{2}\sum_{k=2}^{K-1}\Big(\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi_k) - \Theta^{(2)}_{(\xi_{k+1},\,\xi_k)}(\xi_k)\Big)\phi_2'(\xi_k) \\
&\qquad - \frac{1}{2}\phi_2'(\xi_K)\,\Theta^{(2)}_{(\xi_{K-1},\,\xi_K)}(\xi_K) + \frac{1}{2}\phi_2'(\xi_1)\Theta^{(2)}_{(\xi_1,\,\xi_2)}(\xi_1)\Bigg] \\
&= -c_{1,\alpha}\big(I + II + III + IV + V\big).
\end{aligned} \tag{3.12}$$

For term I, by Taylor's theorem, we obtain

$$\begin{aligned}
|\,I\,| &= \left|\int_0^h \big(\phi_2(\xi) - \phi_2(\xi_1)\big)\xi^{1-\alpha}d\xi\right| \\
&\le Ch \max_{\eta\in[0,h]}|\phi_2'(\eta)|\int_0^h \xi^{1-\alpha}d\xi \le Ch^{3-\frac{\alpha}{2}},
\end{aligned} \tag{3.13}$$

where the last inequality is obtained using Lemma 3.3 (ii) with $n = 1$. For term II, using the definition of $\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}$, we get

$$|\,II_2\,| = \frac{1}{2}\left|\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\phi_2''(\xi)\,\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi)d\xi\right|$$

$$\le C\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\left|\phi_2''(\xi)\right|\left|\int_{\xi_{k-1}}^{\xi}(\xi-y)\,y^{1-\alpha}\,dy+\int_{\xi_k}^{\xi}(\xi-y)\,y^{1-\alpha}\,dy\right|d\xi.$$

A further calculation shows that

$$\begin{aligned}|II_2| &\le C\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\Big(\max_{\xi\in[\xi_{k-1},\,\xi_k]}\left|\phi_2''(\xi)\right|\Big)\Big(h\big(\xi_k^{2-\alpha}-\xi_{k-1}^{2-\alpha}\big)\Big)\\
&\le Ch^2\sum_{k=2}^{K}\Big(\max_{\xi\in[\xi_{k-1},\,\xi_k]}\xi^{\frac{\alpha}{2}-1}\Big)\Big(\xi_k^{2-\alpha}-\xi_{k-1}^{2-\alpha}\Big)\\
&\le Ch^2\sum_{k=2}^{K}\Big(\xi_k^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}\Big)\frac{\xi_k^{1-\frac{\alpha}{2}}+\xi_{k-1}^{1-\frac{\alpha}{2}}}{\xi_{k-1}^{1-\frac{\alpha}{2}}},\end{aligned}$$

where the second inequality is obtained from Lemma 3.3 (ii) with $n=2$. Then, we obtain

$$|II_2|\le Ch^2\sum_{k=2}^{K}\Big(\xi_k^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}\Big)\Big(1+2^{1-\frac{\alpha}{2}}\Big)\le Ch^2. \tag{3.14}$$

To estimate term III, we introduce an auxiliary function

$$G_k(x):=\int_{x}^{\xi_k}(\xi_k-y)y^{1-\alpha}dy,\qquad \text{for } x\in[\xi_{k-1},\,\xi_k].$$

Noticing the definition of $\Theta^{(2)}_{(\xi_{k-1},\xi_k)}$ and G_k, and using Taylor's theorem, we obtain

$$\begin{aligned}|III| &= \frac{1}{2}\left|\sum_{k=2}^{K-1}\Big(\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}(\xi_k)-\Theta^{(2)}_{(\xi_{k+1},\,\xi_k)}(\xi_k)\Big)\phi_2'(\xi_k)\right|\\
&= \frac{1}{2}\left|\sum_{k=2}^{K-1}\Big(G_k(\xi_{k-1})-G_k(\xi_{k+1})\Big)\phi_2'(\xi_k)\right|\\
&\le C\sum_{k=2}^{K-1}\Big(h^3\max_{\eta\in[\xi_{k-1},\,\xi_{k+1}]}\left|G_k'''(\eta)\right|\Big)|\phi_2'(\xi_k)|,\end{aligned}$$

where we have used the fact that $G_k(\xi_k)=G_k'(\xi_k)=0$. After a simple calculation and using Lemma 3.3 (ii) with $m=1$, we further obtain

$$\begin{aligned}|III| &\le C\sum_{k=2}^{K-1}\big(h^3\xi_k^{-\alpha}\big)\xi_k^{\frac{\alpha}{2}}=Ch^2\Big(h\sum_{k=2}^{K-1}\xi_k^{-\frac{\alpha}{2}}\Big)\\
&\le Ch^2\int_0^L\xi^{-\frac{\alpha}{2}}d\xi\le Ch^2.\end{aligned} \tag{3.15}$$

For terms IV, using the definition of $\Theta^{(2)}_{(\xi_{k-1},\,\xi_k)}$ and Lemma 3.3 (ii) with $n=1$, we get

$$\begin{aligned}|IV| &= \frac{1}{2}\left|\phi_2'(\xi_K)\Theta^{(2)}_{(\xi_{K-1},\,\xi_K)}(\xi_K)\right|\\
&= \frac{1}{2}\left|\phi_2'(\xi_K)\int_{\xi_{K-1}}^{\xi_K}(\xi_K-\xi)\xi^{1-\alpha}d\xi\right|\le C\xi_K^{\frac{\alpha}{2}}\big(h^2\max\{\xi_{K-1}^{1-\alpha},\,\xi_K^{1-\alpha}\}\big)\le Ch^2,\end{aligned} \tag{3.16}$$

since $\xi_K=L$ and $\xi_{K-1}=L-h$. Following similar arguments, we obtain the estimates for V as

$$
\begin{aligned}
|V| &= \frac{1}{2}\left|\phi_2'(\xi_1)\Theta^{(2)}_{(\xi_1,\,\xi_2)}(\xi_1)\right| \\
&= \frac{1}{2}\left|\phi_2'(\xi_1)\int_{\xi_2}^{\xi_1}(\xi_1-\xi)\xi^{1-\alpha}d\xi\right| \le C\xi_1^{\frac{\alpha}{2}}\left(h^2\max\{\xi_1^{1-\alpha},\,\xi_2^{1-\alpha}\}\right) \le Ch^{3-\alpha/2}.
\end{aligned}
\tag{3.17}
$$

Combining (3.12)–(3.17) yields the estimate

$$
|e^h_{\alpha,2}(x)| \le C\big(|I|+|II|+|III|+|IV|+|V|\big) \le Ch^2, \qquad \text{for } \alpha\in(0,2).
\tag{3.18}
$$

The error estimate for $\gamma = 1+\frac{\alpha}{2}$ can be done via the estimate on $e^h_{\alpha,2}$. Here, we first rewrite

$$
\phi_\gamma(\xi) = \frac{u(x+\xi)-2u(x)+u(x-\xi)}{\xi^2}\,\xi^{2-\gamma} = \phi_2(\xi)\xi^{2-\gamma}, \qquad \text{for } \gamma\in(\alpha,2].
$$

Then, we can rewrite the error function $e^h_{\alpha,1+\frac{\alpha}{2}}$ as

$$
\begin{aligned}
e^h_{\alpha,\,1+\frac{\alpha}{2}}(x) &= -c_{1,\alpha}\Bigg[\int_{\xi_0}^{\xi_1}\Big(\phi_2(\xi)-\phi_2(\xi_1)\Big)\xi^{1-\alpha}d\xi \\
&\qquad + \sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\Big(\phi_2(\xi)-\frac{\phi_2(\xi_{k-1})+\phi_2(\xi_k)}{2}\Big)\xi^{1-\alpha}d\xi\Bigg] \\
&\qquad - \frac{c_{1,\alpha}}{4}\Bigg[\sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\Big(\phi_2(\xi_{k-1})+\phi_2(\xi_k)\Big)\Big(2\xi^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}-\xi_k^{1-\frac{\alpha}{2}}\Big)\xi^{-\frac{\alpha}{2}}d\xi \\
&\qquad + \sum_{k=2}^{K}\int_{\xi_{k-1}}^{\xi_k}\Big(\xi_k^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}\Big)\Big(\phi_2(\xi_{k-1})-\phi_2(\xi_k)\Big)\xi^{-\frac{\alpha}{2}}d\xi\Bigg] \\
&= e^h_{\alpha,2}(x) - \frac{c_{1,\alpha}}{4}\big(\widetilde{I}+\widetilde{II}\big)
\end{aligned}
\tag{3.19}
$$

with $e^h_{\alpha,2}$ defined in (3.12). It is easy to show that the term $\widetilde{I}$ vanishes, i.e.,

$$
\widetilde{I} = \sum_{k=2}^{K}\Big(\phi_2(\xi_{k-1})+\phi_2(\xi_k)\Big)\int_{\xi_{k-1}}^{\xi_k}\Big(2\xi^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}-\xi_k^{1-\frac{\alpha}{2}}\Big)\xi^{-\frac{\alpha}{2}}d\xi = 0.
\tag{3.20}
$$

For term $\widetilde{II}$, we obtain

$$
\begin{aligned}
|\widetilde{II}| &= \left|\sum_{k=2}^{K}\Big(\xi_k^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}\Big)\big(\phi_2(\xi_{k-1})-\phi_2(\xi_k)\big)\int_{\xi_{k-1}}^{\xi_k}\xi^{-\frac{\alpha}{2}}d\xi\right| \\
&\le C\sum_{k=2}^{K}\big|\phi_2(\xi_{k-1})-\phi_2(\xi_k)\big|\Big(\xi_k^{1-\frac{\alpha}{2}}-\xi_{k-1}^{1-\frac{\alpha}{2}}\Big)^2 \\
&\le Ch^3\sum_{k=2}^{K}\Big(\max_{\zeta\in[\xi_{k-1},\xi_k]}|\phi_2'(\zeta)|\Big)\Big(\max_{\xi\in[\xi_{k-1},\,\xi_k]}\xi^{-\frac{\alpha}{2}}\Big)^2,
\end{aligned}
$$

by Taylor's theorem. Using Lemma 3.3 (ii) with $n=1$, we obtain

$$
\begin{aligned}
|\widetilde{II}| &\le Ch^3\sum_{k=2}^{K}\Big(\max_{\zeta\in[\xi_{k-1},\xi_k]}\zeta^{\frac{\alpha}{2}}\Big)\Big(\max_{\xi\in[\xi_{k-1},\,\xi_k]}\xi^{-\frac{\alpha}{2}}\Big)^2 \\
&\le Ch^2\int_0^L \xi^{-\alpha/2} \le Ch^2.
\end{aligned}
\tag{3.21}
$$

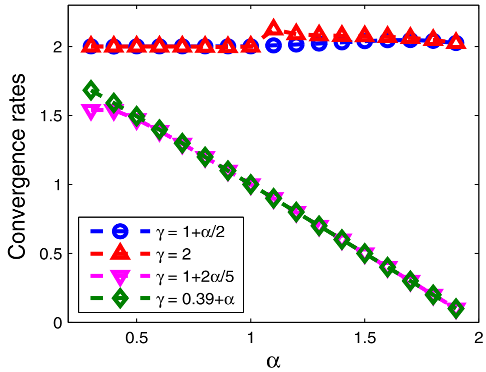

Fig. 2. Convergence rate versus α for different choice of γ, where $u \in C^{3,\alpha/2}(\mathbb{R})$.

Combining (3.18)–(3.21), we get

$$|e^h_{\alpha,1+\frac{\alpha}{2}}(x)| \le C\big(|e^h_{\alpha,2}(x)| + |\widetilde{I}\,| + |\widetilde{II}\,|\big) \le Ch^2, \qquad \text{for } \alpha \in (0,2). \quad \square$$

Theorem 3.2 shows that for $u \in C^{3,\alpha/2}(\mathbb{R})$, if the splitting parameter is chosen as $\gamma = 2$ or $1+\alpha/2$, our numerical method has the accuracy of $\mathcal{O}(h^2)$ uniformly for any $\alpha \in (0,2)$.

Remark 3.2. The results of Theorem 3.2 are consistent with the behavior of the central difference method for the classical Laplace operator. Indeed, for $u \in C^{3,1}(\mathbb{R})$ (corresponding to $u \in C^{3,\alpha/2}(\mathbb{R})$ as $\alpha \to 2$), by Taylor's theorem and mean value theorem, there exist $x^- \in [x-h, x]$ and $x^+ \in [x, x+h]$, such that

$$|e^h| = \big|\phi_2(x,h) - u''(x)\big| = \frac{h}{6}\big|u'''(x^+) - u'''(x^-)\big| \le Ch^2.$$

Remark 3.3. For $u \in C^{3,\alpha/2}(\mathbb{R})$, if the splitting parameter $\gamma \notin \{2, 1+\frac{\alpha}{2}\}$, the accuracy of our method becomes α-dependent. Here, we divide our discussion into two parts: (i). For $1 < \alpha < 2$, we can analytically prove:

$$\big\|(-\Delta)^{\alpha/2}u(x) - (-\Delta)^{\alpha/2}_{h,\gamma}u(x)\big\|_{\infty,\,\Omega} \le Ch^{2-\alpha}, \qquad \gamma \in (\alpha, 2), \text{but } \gamma \ne 1+\tfrac{\alpha}{2}, \tag{3.22}$$

that is, the accuracy is $\mathcal{O}(h^{2-\alpha})$. (ii). For $0 < \alpha \le 1$, it is challenging to obtain a uniform error bound for all $\gamma \in (\alpha, 2)$. In Fig. 2, we provide a numerical study on the convergence rate for different choices of γ. It verifies our analytical results in (3.22) for $\alpha \in (1,2)$. While for $\alpha \in (0,1]$, it shows that the convergence rate is almost $\mathcal{O}(h^{2-\alpha})$, if $\gamma \ne 2$ or $1+\alpha/2$.

Remarks 3.1 and 3.3 show that choosing the splitting parameter $\gamma = 2$ or $1+\frac{\alpha}{2}$ in our method leads to a better accuracy. Hence, in the following sections we will focus our studies on the optimal splitting parameter $\gamma = 2$ and $1+\frac{\alpha}{2}$.

4. Fractional Poisson equation

Compared to the method in [16], our finite difference method proposed in Section 2 has a higher accuracy, but requires lower regularity for the function u. It more closely resembles the central difference scheme for the classical Laplace operator. Moreover, our method is simple to implement, and it can be easily generalized to solve nonlocal problems with the fractional Laplacian. Here, we will apply it to solve the one-dimensional fractional Poisson problem (1.3)–(1.4) and provide its convergence analysis.

We consider (1.3)–(1.4) on the domain $\Omega = (-1,1)$. Let x_i (for $i \in \mathbb{Z}$) denote the uniform grid points on $\mathbb{R}$ with mesh size h. For simplicity of notations, we define

$$\mathcal{S}_\Omega = \{i \mid i \in \mathbb{Z} \text{ and } x_i \in \Omega\}, \qquad \mathcal{S}^c_\Omega = \{i \mid i \in \mathbb{Z} \text{ and } x_i \in \Omega^c\}$$

as the index sets of the grid points in Ω and Ω^c, respectively. Let $\mathbf{u}^h = \{u^h_i\}_{i\in\mathcal{S}_\Omega}$ denote the numerical solution of (1.3)–(1.4) on Ω, with u^h_i representing the numerical approximation of $u(x_i)$. Define the vector $\mathbf{f} = \{f(x_i)\}_{i\in\mathcal{S}_\Omega}$. By applying the scheme in (2.10), we can discretize the fractional Poisson equation (1.3) and write it into matrix-vector form:

$$A\mathbf{u}^h = \mathbf{f}, \tag{4.1}$$

and the boundary condition in (1.4) is discretized as:

$$u^h_i = 0, \qquad \text{for } i \in \mathcal{S}^c_\Omega. \tag{4.2}$$

As discussed previously, A is a symmetric Toeplitz matrix, and the solution $\mathbf{u}^h$ of the difference equations (4.1) can be found by the fast algorithm proposed in [27,4]. In the following, we will study the convergence of the solution of the difference equations (4.1)–(4.2) to that of the fractional Poisson problem (1.3)–(1.4).

As preparation, we let $u_i = u(x_i)$ (for $i \in \mathbb{Z}$) denote a grid function and introduce the maximum norm for the grid function as

$$\|\mathbf{u}\|_{l_\infty(\Omega)} = \max_{i\in\mathcal{S}_\Omega} |u_i|. \tag{4.3}$$

Then, we have the following maximum principle for grid function u_i:

Lemma 4.1. *Suppose that $u_i = u(x_i)$ (for $i \in \mathbb{Z}$) is a grid function satisfying $u_i \equiv 0$ for $i \in \mathcal{S}_\Omega^c$. Then, there is*

$$\begin{cases} \max\limits_{i\in\mathcal{S}_\Omega} u_i \le \max\limits_{i\in\mathcal{S}_\Omega^c} u_i, & \text{if } (-\Delta)_{h,\gamma}^{\alpha/2} u_i \le 0, \ \text{for } i \in \mathcal{S}_\Omega, \\ \min\limits_{i\in\mathcal{S}_\Omega} u_i \ge \min\limits_{i\in\mathcal{S}_\Omega^c} u_i, & \text{if } (-\Delta)_{h,\gamma}^{\alpha/2} u_i \ge 0, \ \text{for } i \in \mathcal{S}_\Omega. \end{cases} \tag{4.4}$$

Proof. We will prove (4.4) by contradiction. If $(-\Delta)_{h,\gamma}^{\alpha/2} u_i \le 0$ for all $i \in \mathcal{S}_\Omega$, we assume that there exists $m \in \mathcal{S}_\Omega$, such that

$$u_m = \max_{i\in\mathcal{S}_\Omega} u_i > \max_{i\in\mathcal{S}_\Omega^c} u_i = 0.$$

On one hand, since $m \in \mathcal{S}_\Omega$, we have

$$(-\Delta)_{h,\gamma}^{\alpha/2} u_m \le 0. \tag{4.5}$$

On the other hand, we obtain from (2.10) that

$$\begin{aligned} (-\Delta)_{h,\gamma}^{\alpha/2} u_m \ge C_{\alpha,\gamma}^h \Bigg[\frac{1}{2} \sum_{\substack{j=1,\\ j\neq m,\, m\pm1}}^{K-1} \frac{(|j-m|+1)^\nu - (|j-m|-1)^\nu}{|j-m|^\gamma} (u_m - u_j) \\ + \frac{2^\nu + \kappa_\gamma - 1}{2} \Big((u_m - u_{m+1}) + (u_m - u_{m-1}) \Big) \\ + \left(\frac{K^\nu - (K-1)^\nu}{K^\gamma} + \frac{2\nu}{\alpha K^\alpha} \right) u_m \Bigg] > 0. \end{aligned} \tag{4.6}$$

The contradiction of (4.5) and (4.6) implies that if $(-\Delta)_{h,\gamma}^{\alpha/2} u_i \le 0$ for all $i \in \mathcal{S}_\Omega$, the global maximum can not be attained inside of the domain Ω.

The proof of the other case follows along similar lines, by replacing u_i with $-u_i$. □

Lemma 4.2. *Define a function*

$$v(x) = \begin{cases} 5 - x^2, & \text{if } x \in \Omega \\ 0, & \text{if } x \in \Omega^c. \end{cases} \tag{4.7}$$

Then for any $\alpha \in (0, 2)$, the grid function $v_i = v(x_i)$ satisfies

$$(-\Delta)_{h,\nu}^{\alpha/2} v_i > 1, \quad \text{for} \quad i \in \mathcal{S}_\Omega, \tag{4.8}$$

$$v_i = 0, \quad \text{for} \quad i \in \mathcal{S}_\Omega^c. \tag{4.9}$$

Proof. Here, we focus on the proof of (4.8), since (4.9) is straightforward from the definition of $v(x)$. First, we have the following discussion on the function $\phi_\gamma(x, \xi)$. For $x \in \Omega$ and $\xi > 0$,

(i) If both $x \pm \xi \in \Omega$, then

$$\phi_\gamma(x,\xi) = \begin{cases} \dfrac{v(x+\xi) - 2v(x) + v(x-\xi)}{\xi^{1+\alpha/2}} = -2\xi^{1-\alpha/2}, & \text{if } \gamma = 1 + \frac{\alpha}{2}, \\ \dfrac{v(x+\xi) - 2v(x) + v(x-\xi)}{\xi^2} = -2, & \text{if } \gamma = 2. \end{cases}$$

(ii) If either $x + \xi \in \Omega^c$ or $x - \xi \in \Omega^c$, then

$$\phi_\gamma(x,\xi) < \begin{cases} -2\xi^{1-\alpha/2}, & \text{if } \gamma = 1 + \frac{\alpha}{2}, \\ -2, & \text{if } \gamma = 2, \end{cases}$$

since $v(x \pm \xi)/\xi^{1+\alpha/2} \ge 0$ or $v(x \pm \xi)/\xi^2 \ge 0$, for any $x \in \Omega$ and $\xi > 0$.

For both $\gamma = 1 + \frac{\alpha}{2}$ and $\gamma = 2$, we obtain the following estimate from (2.9):

$$(-\Delta)_{h,\gamma}^{\alpha/2} v_i > 2c_{1,\alpha}\left[\frac{1}{2-\alpha}\sum_{k=1}^{K}\left(\xi_k^{1-\alpha/2} - \xi_{k-1}^{1-\alpha/2}\right)\left(\xi_k^{1-\alpha/2} + \xi_{k-1}^{1-\alpha/2}\right) + \frac{1}{\alpha L^\alpha} v_i\right]$$
$$= 2c_{1,\alpha}\left[\frac{1}{2-\alpha}\sum_{k=1}^{K}\left(\xi_k^{2-\alpha} - \xi_{k-1}^{2-\alpha}\right) + \frac{1}{\alpha L^\alpha} v_i\right]$$
$$= 2c_{1,\alpha}\left(\frac{L^{2-\alpha}}{2-\alpha} + \frac{1}{\alpha L^\alpha} v_i\right), \qquad \text{for } i \in \mathcal{S}_\Omega.$$

Since $L = 2$, and for any $i \in \mathcal{S}_\Omega$, there is $v_i > 4$, we then obtain

$$(-\Delta)_{h,\gamma}^{\alpha/2} v_i > c_{1,\alpha}\, 2^{3-\alpha}\left(\frac{1}{2-\alpha} + \frac{1}{\alpha}\right) = \frac{4}{\sqrt{\pi}}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})},$$

by the definition of the constant $c_{1,\alpha}$. Notice that

$$\min_{\alpha\in(0,2)} \Gamma\left(\frac{1+\alpha}{2}\right) > \frac{4}{5}, \qquad \max_{\alpha\in(0,2)} \Gamma\left(2 - \frac{\alpha}{2}\right) < 1.$$

Hence, we obtain

$$(-\Delta)_{h,\gamma}^{\alpha/2} v_i > \frac{16}{5\sqrt{\pi}},$$

which leads to (4.8) immediately. □

Lemma 4.3. *Suppose that $u_i = u(x_i)$ (for $i \in \mathbb{Z}$) is a grid function satisfying $u_i \equiv 0$ for $i \in \mathcal{S}_\Omega^c$. Then, there is*

$$\|\mathbf{u}\|_{l_\infty(\Omega)} \le 5\left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)}. \tag{4.10}$$

Proof. Introduce the grid function

$$w_i = u_i - \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)} v_i, \qquad \text{for } i \in \mathbb{Z}, \tag{4.11}$$

where v_i is defined in Lemma 4.2. Then, we obtain

$$(-\Delta)_{h,\gamma}^{\alpha/2} w_i = (-\Delta)_{h,\gamma}^{\alpha/2} u_i - \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)}\left[(-\Delta)_{h,\gamma}^{\alpha/2} v_i\right]$$
$$\le \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)}\left[1 - (-\Delta)_{h,\gamma}^{\alpha/2} v_i\right] \le 0, \qquad i \in \mathcal{S}_\Omega$$

by using (4.3) and (4.8). From Lemma 4.1, we then get

$$\max_{i\in\mathcal{S}_\Omega} w_i \le \max_{i\in\mathcal{S}_\Omega^c} w_i = 0,$$

i.e., $w_i \le 0$ for all $i \in \mathcal{S}_\Omega$. Noticing the definition of w_i in (4.11), we obtain

$$u_i \le \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)} v_i, \qquad \text{for } i \in \mathcal{S}_\Omega. \tag{4.12}$$

On the other hand, we introduce the function

$$z_i = u_i + \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)} v_i. \qquad \text{for } i \in \mathbb{Z}. \tag{4.13}$$

Similarly, we can obtain

$$(-\Delta)_{h,\gamma}^{\alpha/2} z_i = (-\Delta)_{h,\gamma}^{\alpha/2} u_i + \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)}\left[(-\Delta)_{h,\gamma}^{\alpha/2} v_i\right]$$
$$\ge (-\Delta)_{h,\gamma}^{\alpha/2} u_i + \left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)} \ge 0, \qquad i \in \mathcal{S}_\Omega$$

Then from Lemma 4.1, we get

$$\min_{i\in\mathcal{S}_\Omega} z_i \ge \min_{i\in\mathcal{S}_\Omega^c} z_i = 0,$$

i.e., $z_i \ge 0$ for all $i \in \mathcal{S}_\Omega$. Noticing the definition of z_i in (4.13), we obtain

$$u_i \ge -\left\|(-\Delta)_{h,\gamma}^{\alpha/2}\mathbf{u}\right\|_{l_\infty(\Omega)} v_i, \qquad \text{for } i \in \mathcal{S}_\Omega. \tag{4.14}$$

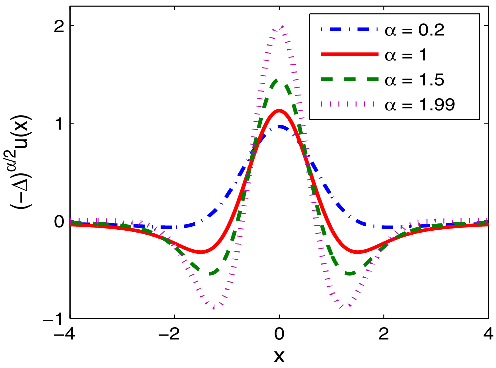

Fig. 3. Numerical results of $(-\Delta)^{\alpha/2}u$ with u defined in (5.1), where the displayed domain is much smaller than our computational domain.

Combining (4.12) and (4.14), we obtain

$$|u_i| \le \left\|(-\Delta)^{\alpha/2}_{h,\gamma}\mathbf{u}\right\|_{l_\infty(\Omega)}|v_i|,$$

which implies (4.10) immediately, since $\|\mathbf{v}\|_{l_\infty(\Omega)} = 5$. $\square$

Then, we have the following theorem for the numerical solution of the fractional Poisson equation (4.1)–(4.2):

Theorem 4.1. *Suppose that u is the exact solution of the fractional Poisson equation (1.3)–(1.4), and it satisfies the same conditions as in Theorem 3.1 or 3.2. Let $\mathbf{u}^h$ be the solution of the difference equations (4.1)–(4.2). Then, there is*

$$\left\|\mathbf{u} - \mathbf{u}^h\right\|_{l_\infty(\Omega)} \le 5\left\|(-\Delta)^{\alpha/2}\mathbf{u} - (-\Delta)^{\alpha/2}_{h,\gamma}\mathbf{u}\right\|_{l_\infty(\Omega)}. \tag{4.15}$$

Proof. The proof of (4.15) can be done by using Lemma 4.3 to $\mathbf{u} - \mathbf{u}^h$. $\square$

Theorem 4.1 implies that the numerical errors in solving the fractional Poisson equation (1.3)–(1.4) are bounded by those in discretizing the fractional Laplacian. Therefore, from Theorem 3.1 (resp. 3.2), we can conclude that if the solution of (1.3)–(1.4) satisfies $u \in C^{3,\alpha/2}(\mathbb{R})$ (resp. $u \in C^{1,\alpha/2}(\mathbb{R})$), our method has the accuracy of *at least* $\mathcal{O}(h^2)$ (resp. $\mathcal{O}(h^{1-\alpha/2})$). The error estimate in Theorem 4.1 relies fundamentally on the smoothness of the solution u, which raises the question of determining conditions under which the desired regularity can be attained. Unlike for integer-order elliptic equations, the regularity of the fractional Poisson problem's solution cannot be guaranteed by means of a sufficiently smooth source term f and boundary (see [18,31,32]). One possible numerical recourse for maintaining the accuracy of our method in the presence of limited regularity would be to identify spatial regions in which the solution displays said lack of smoothness and to increase the spatial resolution there. In [22] it was shown that for sufficiently smooth problems, this deterioration of regularity arises exclusively near the domain's boundary $\partial\Omega$. In fact, higher order Hölder estimates for u can be obtained for sufficiently smooth problems away from the boundary, i.e. in sets of the form $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \rho\}$, where $\rho > 0$. The cost of using non-uniform meshes includes the break down of the stiffness matrix's Toeplitz structure.

5. Numerical experiments

In this section, we study the accuracy of our method in discretizing the fractional Laplacian $(-\Delta)^{\alpha/2}$ and in solving the fractional Poisson problem (1.3)–(1.4). We will compare our method with the scheme proposed in [16], which is the current state of the art for directly discretizing the fractional Laplacian. Unless otherwise stated, the splitting parameter is chosen as $\gamma = 1 + \alpha/2$ in our simulations.

5.1. Discretization of the fractional Laplacian

Example 5.1.1. We compare our method with the scheme proposed in [16]. For easy comparison, we use the same example as that in [16, Section 6.1], where a Gaussian function

$$u(x) = e^{-x^2}, \qquad x \in \mathbb{R} \tag{5.1}$$

is considered, i.e., $u \in C^\infty(\mathbb{R})$. Since u exponentially decays as $|x| \to \infty$, in practice one can truncate it into a bounded domain $(-l, l)$ with l sufficiently large and approximate $u(x) = 0$ for $x \in \mathbb{R}\backslash(-l, l)$. Here, we choose $l = 10$. Fig. 3 demonstrates the numerical results of $(-\Delta)^{\alpha/2}u$ for various α. It shows that the function $(-\Delta)^{\alpha/2}u$ decays to zero as $|x| \to \infty$; the

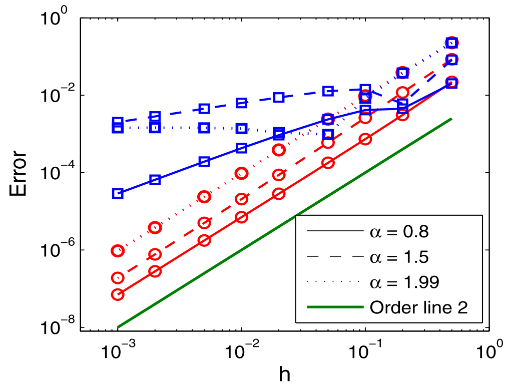

Fig. 4. Numerical errors in computing $(-\Delta)^{\alpha/2}u(0)$ of our method ('○') and the method in [16] ('□'). It shows that our method has a second order convergence rate for any $\alpha \in (0, 2)$.

larger the fractional parameter α is, the faster the solution decays. We find that as $\alpha \to 2$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ converges to the standard Laplace operator $-\Delta$ (see Fig. 3 for $\alpha = 1.99$).

Next we compare our method with the scheme in [16] in approximating the value $(-\Delta)^{\alpha/2}u(0)$. At $x = 0$, the exact solution can be computed as:

$$(-\Delta)^{\alpha/2}u(0) = (-\Delta)^{\alpha/2}u(x)\,|_{x=0} = \frac{2^\alpha}{\sqrt{\pi}}\Gamma\Big(\frac{1+\alpha}{2}\Big).$$

In Fig. 4, we present the numerical errors $|(-\Delta)^{\alpha/2}u(0) - (-\Delta)_h^{\alpha/2}u(0)|$ of both methods, where an order line in included for easy comparison. It shows that for each $\alpha \in (0, 2)$, when a small mesh size h is used, the errors of our method are much smaller than those from the method in [16]. The accuracy of our method is $\mathcal{O}(h^2)$, independent of the parameter α. In contrast, the convergence rate of the method in [16] greatly depends on the power α, e.g., $\mathcal{O}(h^{2-\alpha})$ in Fig. 4, as a linear interpolation is used. Hence, as $\alpha \to 2$, this method converges extremely slowly. It is mentioned in [16] that this accuracy could be improved to $\mathcal{O}(h^{3-\alpha})$, if quadratic interpolation is used. However, using a quadratic interpolation significantly complicates the computer implementation, and the convergence rate for $\alpha > 1$ is still less than second order.

Example 5.1.2. We consider a function u of the form:

$$u(x) = \begin{cases} (1-x^2)^{s+\frac{\alpha}{2}}, & \text{for } x \in \Omega = (-1, 1), \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}, \tag{5.2}$$

for $s \in \mathbb{N}$. It is easy to verify that $u(x) \in C^{s,\alpha/2}(\mathbb{R})$, and it has compact support on $(-1, 1)$. The fractional Laplacian of $u(x)$ can be found exactly [13,12]:

$$(-\Delta)^{\alpha/2}u(x) = \frac{2^\alpha\Gamma(\frac{\alpha+1}{2})\Gamma(s+1+\frac{\alpha}{2})}{\sqrt{\pi}\,\Gamma(s+1)}\,{}_2F_1\left(\frac{\alpha+1}{2}, -s; \frac{1}{2}; x^2\right), \qquad x \in (-1, 1),$$

where ${}_2F_1$ denotes the Gauss hypergeometric function. In this example, we will test the accuracy of our method for different s.

Tables 1 and 2 present numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)_{h,\gamma}^{\alpha/2}u\|_{l_\infty(\Omega)}$ and convergence rates of our method for various α, where u is defined in (5.2) with $s = 1$ and $s = 3$, respectively. We find that for the same mesh size h, the larger the parameter α, the bigger the numerical errors. For $s = 1$, Table 1 shows that the convergence rates of our method is $\mathcal{O}(h^{1-\alpha/2})$ for any $\alpha \in (0, 2)$, which confirms our analytical results in Theorem 3.1. While for $s = 3$, our method has accuracy of $\mathcal{O}(h^2)$ for any $\alpha \in (0, 2)$; see Table 2. This observation is consistent with our conclusion in Theorem 3.2.

Additionally, we compare the accuracy of our method with that in [16] in discretizing the fractional Laplacian. Fig. 5 displays numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)_{h,\gamma}^{\alpha/2}u\|_{l_\infty(\Omega)}$ versus mesh size h, for various cases. From Fig. 5 a), we find that if the function $u \in C^{1,\alpha/2}(\mathbb{R})$, both methods have the same accuracy of $\mathcal{O}(h^{1-\alpha/2})$, but for fixed α and h the numerical errors of our method are much smaller. While for smooth enough function u, our method has second order of accuracy for any $\alpha \in (0, 2)$; in contrast, the accuracy of the method in [16] is $\mathcal{O}(h^{2-\alpha})$; see Fig. 5 c) & d).

Remark 5.1. For $u \in C^{2,\alpha/2}(\mathbb{R})$, we study the convergence rates of our finite difference method numerically; see Fig. 6. It shows that for $\alpha \le 0.7$, the convergence rate in l_∞-norm is $\mathcal{O}(h^2)$. While for $\alpha > 0.7$, the convergence rate decreases toward 1, which is better than the case for $u \in C^{1,\alpha/2}(\mathbb{R})$. In fact, this behavior is consistent with that of the central difference scheme to the classical Laplace operator – for $u \in C^{2,1}(\mathbb{R})$, by Taylor's theorem and mean value theorem, there exist $x^- \in [x-h, x]$ and $x^+ \in [x, x+h]$, such that

Table 1
Numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)_{h,\gamma}^{\alpha/2}u\|_{l_\infty(\Omega)}$ and convergence rates of our method, where u is defined in (5.2) with $s=1$, i.e., $u \in C^{1,\alpha/2}(\mathbb{R})$. The symbol 'c.r.' represents convergence rate.

α \ h	1/32	1/64	1/128	1/256	1/512	1/1024	1/2048
0.2	4.846E-4	2.629E-4	1.417E-4	7.615E-5	4.086E-5	2.191E-5	1.174E-5
	c.r.	0.8822	0.8917	0.8961	0.8981	0.8991	0.8996
0.6	1.025E-3	6.497E-4	4.046E-4	2.502E-4	1.543E-4	9.505E-5	5.852E-5
	c.r.	0.6582	0.6834	0.6934	0.6974	0.6990	0.6996
1	2.291E-3	1.544E-3	1.071E-3	7.516E-4	5.297E-4	3.740E-4	2.643E-4
	c.r.	0.5691	0.5278	0.5113	0.5047	0.5020	0.5009
1.5	2.460E-2	2.015E-2	1.675E-2	1.401E-2	1.176E-2	9.874E-3	8.298E-3
	c.r.	0.2885	0.2664	0.2573	0.2534	0.2516	0.2508
1.9	2.891E-2	2.632E-2	2.487E-2	2.381E-2	2.291E-2	2.209E-2	2.132E-2
	c.r.	0.1352	0.0817	0.0628	0.0556	0.0526	0.0513

Table 2
Numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)_{h,\gamma}^{\alpha/2}u\|_{l_\infty(\Omega)}$ and convergence rates of our method, where u is defined in (5.2) with $s=3$, i.e., $u \in C^{3,\alpha/2}(\mathbb{R})$. The symbol 'c.r.' represents convergence rate.

α \ h	1/32	1/64	1/128	1/256	1/512	1/1024	1/2048
0.2	5.676E-5	1.418E-5	3.544E-6	8.861E-7	2.215E-7	5.538E-8	1.384E-8
	c.r.	2.0009	2.0003	2.0001	2.0000	2.0000	2.0001
0.6	2.230E-4	5.550E-5	1.385E-5	3.461E-6	8.651E-7	2.162E-7	5.406E-8
	c.r.	2.0062	2.0024	2.0009	2.0003	2.0001	2.0000
1	5.643E-4	1.379E-4	3.414E-5	8.490E-6	2.117E-6	5.290E-7	1.321E-7
	c.r.	2.0324	2.0147	2.0074	2.0037	2.0007	2.0011
1.5	2.742E-3	8.050E-4	2.104E-4	5.251E-5	1.286E-5	3.129E-6	7.606E-7
	c.r.	1.7683	1.9356	2.0027	2.0297	2.0391	2.0405
1.9	1.450E-2	4.273E-3	1.140E-3	2.910E-4	7.275E-5	1.801E-5	4.436E-6
	c.r.	1.7631	1.9062	1.9699	1.9999	2.0143	2.0212

$$|e^h| = \left|\phi_2(x,h) - u''(x)\right| = \frac{1}{2}\left|\left(u''(x^+) - u(x)\right) + \left(u''(x^-) - u(x)\right)\right| \le Ch.$$

5.2. Solution of the fractional Poisson equation

In this section, we apply our numerical method to solve the fractional Poisson equation with extended homogeneous Dirichlet boundary conditions:

$$(-\Delta)^{\alpha/2}u(x) = f(x), \qquad x \in (-1,1), \tag{5.3}$$

$$u(x) = 0, \qquad x \in \mathbb{R}\backslash(-1,1). \tag{5.4}$$

We will study the accuracy of our method for various functions f.

Example 5.2.1. For easy comparison, here we choose the function

$$f(x) = \frac{2^\alpha \Gamma(\frac{\alpha+1}{2})\Gamma(s+1+\frac{\alpha}{2})}{\sqrt{\pi}\,\Gamma(s+1)}\, {}_2F_1\left(\frac{\alpha+1}{2}, -s; \frac{1}{2}; x^2\right), \qquad x \in (-1,1), \tag{5.5}$$

for $s \in \mathbb{N}$. Then, the exact solution of (5.3)–(5.4) is given by (5.2).

Tables 3 and 4 present the numerical errors $\|u - u_h\|_{l_\infty(\Omega)}$ and convergence rates of our method for $s=1$ and 3 in (5.5), respectively. We find that the convergence rate of our method in Table 3 is at least $\mathcal{O}(h^{1+\alpha/2})$. This implies that for the same condition $u \in C^{1,\alpha/2}(\mathbb{R})$, our method is more accurate in solving the fractional Poisson problem than in discretizing the fractional Laplacian $(-\Delta)^{\alpha/2}$, consistent with the central difference scheme for the classical Poisson problem. Table 4 shows that for solution $u \in C^{3,\alpha/2}(\mathbb{R})$, the convergence rate is $\mathcal{O}(h^2)$, same as that in Table 2 for discretizing the fractional Laplacian. The above observations are consistent with our theoretical results in Theorem 4.1.

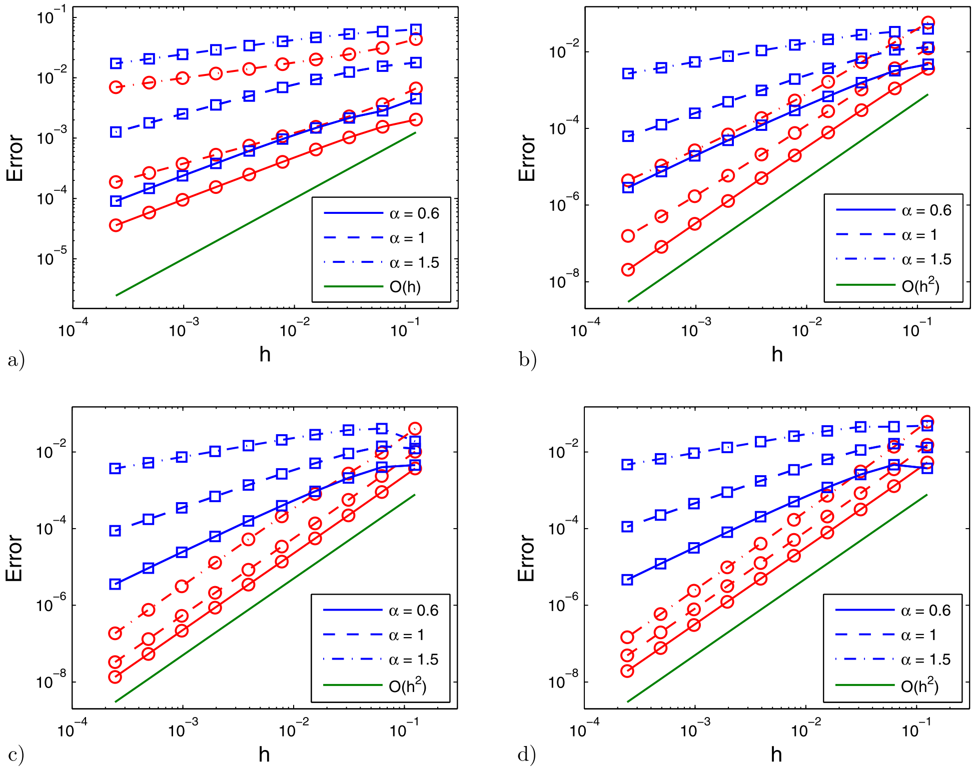

Fig. 5. Numerical errors $\|(-\Delta)^{\alpha/2}u - (-\Delta)^{\alpha/2}_{h,\gamma}u\|_{l_\infty(\Omega)}$ of our method ('○') and the method in [16] ('□'), with $u \in C^{s,\alpha/2}(\mathbb{R})$ defined in (5.2), where a) $s = 1$; b) $s = 2$; c) $s = 3$; d) $s = 4$.

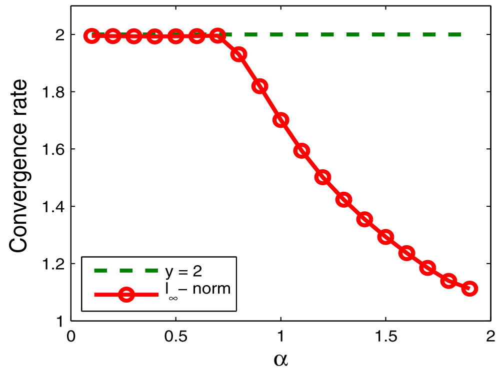

Fig. 6. Convergence rate versus α, where $u \in C^{2,\alpha/2}(\mathbb{R})$ is defined in (5.2) with $s = 2$.

Furthermore, we show in Fig. 7 the convergence rates of our method, where the fitting lines (e.g., $y = 1 + \alpha/2$ or $y = 2$) are present for the sake of easy comparison. From Fig. 7 a), we find that the accuracy is $\mathcal{O}(h^{1+\alpha/2})$ for $\alpha \in (0, 1.5]$, but for $\alpha > 1.5$ it becomes $\mathcal{O}(h^2)$. While if the solution u is smooth enough, the convergence rate is always $O(h^2)$, for any $\alpha \in (0, 2)$ (see Fig. 7 b)).

Example 5.2.2. We choose the function $f(x) \equiv 1$ for $x \in (-1, 1)$. Then, the exact solution of (5.3)–(5.4) is given by [12]:

$$u(x) = \begin{cases} -\dfrac{1}{\Gamma(1+\alpha)}(1-x^2)^{\alpha/2}, & \text{for} \quad x \in (-1, 1), \\ 0, & \text{for} \quad x \in \mathbb{R}\backslash(-1, 1). \end{cases} \tag{5.6}$$

It is easy to show that the solution $u \in C^{0,\alpha/2}(\mathbb{R})$, which has lower regularity than the solutions in Example 5.2.1. For $u \in C^{0,\alpha/2}(\mathbb{R})$, although our method in discretizing $(-\Delta)^{\alpha/2}u$ does not converge, it works well in finding the solution of the fractional Poisson problem (5.3)–(5.4). This phenomenon is consistent with that of the central finite difference method for solving the classical Poisson problem.

Table 3
Numerical errors $\|u - u_h\|_{l_\infty(\Omega)}$ and convergence rates of our method in solving (5.3)–(5.4), where f is defined in (5.5) with $s = 1$. The symbol 'c.r.' represents convergence rate.

α \ h	1/16	1/32	1/64	1/128	1/256	1/512	1/1024
0.1	3.612E-4	1.786E-4	8.723E-5	4.237E-5	2.052E-5	9.924E-6	4.796E-6
	c.r.	1.0163	1.0336	1.0419	1.0460	1.0480	1.0490
0.6	3.263E-4	1.541E-4	6.698E-5	2.810E-5	1.160E-5	4.746E-6	1.935E-6
	c.r.	1.0827	1.2018	1.2531	1.2771	1.2887	1.2944
1	4.826E-4	1.187E-4	2.937E-5	7.534E-6	2.501E-6	8.556E-7	2.974E-7
	c.r.	2.0240	2.0144	1.9629	1.5908	1.5476	1.5244
1.4	1.215E-3	2.932E-4	7.139E-5	1.751E-5	4.443E-6	1.348E-6	4.120E-7
	c.r.	2.0506	2.0384	2.0278	1.9783	1.7205	1.7103
1.9	3.179E-3	7.770E-4	1.899E-4	4.641E-5	1.135E-5	2.780E-6	6.803E-7
	c.r.	2.0327	2.0330	2.0324	2.0313	2.0301	2.0307

Table 4
Numerical errors $\|u - u_h\|_{l_\infty(\Omega)}$ and convergence rates of our method in solving (5.3)–(5.4), where f is defined in (5.5) with $s = 3$. The symbol 'c.r.' represents convergence rate.

α \ h	1/16	1/32	1/64	1/128	1/256	1/512	1/1024
0.1	9.746E-5	2.432E-5	6.077E-6	1.519E-6	3.797E-7	9.494E-8	2.373E-8
	c.r.	2.0027	2.0007	2.0002	2.0001	2.0000	2.0000
0.6	4.294E-4	1.063E-4	2.647E-5	6.609E-6	1.652E-6	4.128E-7	1.032E-7
	c.r.	2.0143	2.0053	2.0020	2.0007	2.0003	2.0001
1	6.232E-4	1.513E-4	3.725E-5	9.240E-6	2.301E-6	5.741E-7	1.434E-7
	c.r.	2.0424	2.0220	2.0113	2.0057	2.0029	2.0015
1.4	8.925E-4	2.083E-4	4.955E-5	1.197E-5	2.921E-6	7.186E-7	1.777E-7
	c.r.	2.0990	2.0717	2.0502	2.0344	2.0232	2.0155
1.9	2.037E-3	4.916E-4	1.183E-4	2.850E-5	6.869E-6	1.657E-6	3.996E-7
	c.r.	2.0511	2.0545	2.0541	2.0527	2.0512	2.0521

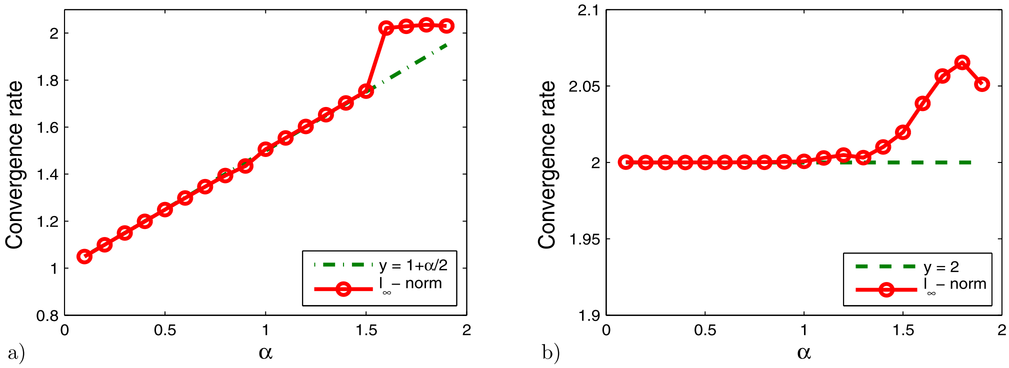

Fig. 7. Convergence rates of our method in solving (5.3)–(5.4), where f is defined in (5.5) with a) $s = 1$ or (b) $s = 3$.

Table 5 displays the numerical errors $\|u - u_h\|_{l_\infty(\Omega)}$ and convergence rates of our method. Compared to the results in Tables 3 and 4, in this case our method converges slower as the mesh size h reduces, since the solution is less smooth. Our observations show that the convergence rate is $\mathcal{O}(h^{\alpha/2})$. Plot 8 compares the convergence rate of our method, for different α. It shows that our method has an accuracy of $\mathcal{O}(h^{\alpha/2})$ for any $\alpha \in (0, 2)$, consistent with our observations in Table 5.

6. Conclusion

We proposed a novel and accurate finite difference method based on the weighted trapezoidal rule to discretize the fractional Laplacian. The novelty of our method is that we formulated the fractional Laplacian as the weighted integral of

Table 5
Numerical errors $\|u - u_h\|_{l_\infty(\Omega)}$ and convergence rates of our method in solving (5.3)–(5.4), where $f(x) \equiv 1$ for $x \in (-1, 1)$. The symbol 'c.r.' represents convergence rate.

α \ h	1/16	1/32	1/64	1/128	1/256	1/512	1/1024
0.2	4.516E-2	4.162E-2	3.860E-2	3.591E-2	3.346E-2	3.119E-2	2.909E-2
	c.r.	0.1176	0.1087	0.1043	0.1022	0.1011	0.1005
0.6	6.667E-2	5.365E-2	4.337E-2	3.515E-2	2.851E-2	2.315E-2	1.880E-2
	c.r.	0.3137	0.3068	0.3034	0.3017	0.3008	0.3004
1	4.500E-2	3.158E-2	2.225E-2	1.570E-2	1.109E-2	7.841E-3	5.543E-3
	c.r.	0.5108	0.5054	0.5027	0.5013	0.5007	0.5003
1.5	1.501E-3	8.871E-3	5.258E-3	3.122E-3	1.855E-3	1.102E-3	6.553E-4
	c.r.	0.7591	0.7546	0.7523	0.7511	0.7506	0.7503
1.9	2.060E-3	1.054E-3	5.427E-4	2.801E-4	1.448E-4	7.489E-4	3.875E-5
	c.r.	0.9663	0.9581	0.9541	0.9521	0.9510	0.9505

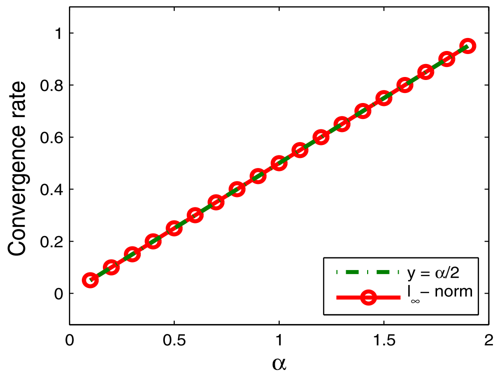

Fig. 8. Convergence rates of our method in solving (5.3)–(5.4) with $f(x) \equiv 1$ for $x \in (-1, 1)$.

a weak singular function, so as to avoid directly discretizing the hypersingular integral. Our method, closely resembling the central difference scheme for the classical Laplace operator, has higher accuracy than other existing methods in the literature. The accuracy of our method was studied analytically and numerically for discretizing the fractional Laplacian and for solving the fractional Poisson equation.

Error estimates were provided to understand the accuracy of our method. For function $u \in C^{1,\alpha/2}(\mathbb{R})$, we proved that our method has an accuracy of $\mathcal{O}(h^{1-\alpha/2})$ with h a small mesh size, for any splitting parameter $\gamma \in (\alpha, 2]$. Additionally, our numerical studies showed that although different choices of γ leads to the same convergence rate, numerical errors are usually smaller when choosing $\gamma = 1 + \frac{\alpha}{2}$ or 2. For function $u \in C^{3,\alpha/2}(\mathbb{R})$, we proved that our method with $\gamma = 2$ or $1 + \frac{\alpha}{2}$ has an accuracy of $\mathcal{O}(h^2)$, independent of the power $\alpha \in (0, 2)$. Extensive numerical investigations were carried out to compare our method with the finite difference method proposed in [16]. We showed that for less smooth function, both methods have the same convergence rate, but the numerical errors of our method are much smaller. While for sufficiently smooth function, our method has an accuracy of $\mathcal{O}(h^2)$, considerably better than $\mathcal{O}(h^{2-\alpha})$ – the accuracy of the method in [16].

Our method was applied to solve the fractional Poisson equation. We proved that the numerical errors in the solution of the fractional Poisson equation are bounded by the errors in discretizing the fractional Laplacian. Furthermore, numerical simulations showed that under the same condition of u, our method has a higher accuracy in solving the fractional Poisson equation than that in discretizing the fractional Laplacian, consistent with the behavior of the well-known central difference method for the classical Poisson problem.

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